Institut für Angewandte Mathematik
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“Markov Processes”, Problem Sheet 3

Please hand in your solutions by Friday, October 31, 13.00.

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1. (Reduction to the time-homogeneous case). Suppose that $((X_t)_{t\in\mathbb{Z}_+},\mathbb{P})$ is a Markov chain with state space $(S,\mathcal{B})$ and one step transition kernels π_t, $t\in\mathbb{N}$.

a) Determine the transition kernel and the generator of the time-space process (t,X_t).

b) Conclude that for any function $f\in\mathcal{F}_b(\mathbb{Z}_+\times S)$, the process

$$M_t^{[f]} \;=\; f(t,X_t)-\sum_{k=0}^{t-1}\mathcal{L}_k(f(k+1,\cdot))(X_k)-\sum_{k=0}^{t-1}(f(k+1,X_k)-f(k,X_k))$$

is a martingale, where $(\mathcal{L}_t)$ are the generators of (X_t).

c) What would be a corresponding statement in continuous time (without proof)?

2. (Brownian motion reflected at 0). Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion defined on a probability space $(\Omega,\mathfrak{A},\mathbb{P})$.

a) Show that $X_t=|B_t|$ is a Markov process with transition density

$$p_t^{\mathrm{refl}}(x,y) \;=\; \frac{1}{\sqrt{2\pi t}}\left(\exp\left(-\frac{(y-x)^2}{2t}\right)+\exp\left(-\frac{(y+x)^2}{2t}\right)\right).$$

b) Prove that $(X_t,\mathbb{P})$ solves the martingale problem for the operator $\mathcal{L}f=\frac{1}{2}f''$ with domain

$$\mathcal{A} \;=\; \{f\in C_b^2([0,\infty)) : f'(0)=0\}.$$

Hint: Note that functions in $\mathcal{A}$ can be extended to symmetric functions in $C_b^2(\mathbb{R})$.

c) Construct another solution to the martingale problem for $\mathcal{L}$ with domain $C_0^\infty(0,\infty)$. Does it also solve the martingale problem in b)?

3. (Strong Markov property and Harris recurrence). Let $(X_n, \mathbb{P}_x)$ be a time homogeneous $(\mathcal{F}_n)$ Markov chain on the state space $(S, \mathcal{B})$ with transition kernel $\pi(x, \mathrm{d}y)$.

a) Show that if T is a finite $(\mathcal{F}_n)$ stopping time, then conditionally given $\mathcal{F}_T$, the process $\hat{X}_n := X_{T+n}$ is a Markov chain with transition kernel π starting in X_T.

b) Conclude that a set $A \in \mathcal{B}$ is *Harris recurrent*, i.e.,

$$\mathbb{P}_x[X_n \in A \text{ for some } n \geq 1] = 1 \quad \text{for any } x \in A,$$

if and only if

$$\mathbb{P}_x[X_n \in A \text{ infinitely often}] = 1 \quad \text{for any } x \in A.$$

4. (Strong Markov property in continuous time). Suppose that $(X_t, \mathbb{P}_x)$ is a time homogeneous $(\mathcal{F}_t)$ Markov process in continuous time with state space $\mathbb{R}^d$ and transition semigroup (p_t).

a) Let T be an $(\mathcal{F}_t)$ stopping time taking only the discrete values $t_i = ih$, $i \in \mathbb{Z}_+$, for some fixed $h \in (0, \infty)$. Prove that for any initial value $x \in \mathbb{R}^d$ and any non-negative measurable function $F \colon (\mathbb{R}^d)^{[0,\infty)} \to \mathbb{R}$,

$$\mathbb{E}_x \left[F(X_{T+\bullet}) \mid \mathcal{F}_T \right] \;=\; \mathbb{E}_{X_T}[F(X)] \qquad \mathbb{P}_x\text{-almost surely.} \tag{1}$$

b) The transition semigroup (p_t) is called *Feller* iff for every $t \geq 0$ and every bounded continuous function $f \colon \mathbb{R}^d \to \mathbb{R}$, $x \mapsto (p_t f)(x)$ is continuous. Prove that if $t \mapsto X_t(\omega)$ is right continuous for all ω and (p_t) is a Feller semigroup, then for *every* $(\mathcal{F}_t)$ stopping time $T \colon \Omega \to [0, \infty)$,

$$\mathbb{E}_x \left[f(X_{T+t}) \mid \mathcal{F}_T \right] \;=\; \mathbb{E}_{X_T}[f(X_t)] \tag{2}$$

holds $\mathbb{P}_x$-almost surely for all $t \geq 0$ and all $f \in C_b(\mathbb{R}^d)$.

c) Conclude that, under the assumptions of b), the strong Markov property (1) holds for every $(\mathcal{F}_t)$ stopping time $T \colon \Omega \to [0, \infty)$.