Institut für Angewandte Mathematik
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„Markov Processes", Problem Sheet 2

Please hand in your solutions until Friday, October 24, 13.00.

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1. (Interacting particle systems). Suppose that (V, E) is a finite graph, and let

$$S \;=\; \{0,1\}^V \;=\; \{\eta = (\eta_i)_{i\in V} \,:\, \eta_i \in \{0,1\} \text{ for all } i \in V\}.$$

We consider the following discrete time stochastic processes with state space S:

(i) *Voter model:* In each step, a vertex $i \in V$ and a neighbouring vertex $j \in V$ are chosen uniformly at random, and the value of η_i is changed to that of η_j. All other components of η stay unchanged.

(ii) *Contact process:* In each step, a vertex $i \in V$ and a neighbouring vertex $j \in V$ are chosen uniformly at random. If $\eta_i = 1$ then η_i is updated to 0 with probability p and to 1 with probability $1-p$, where $p \in (0,1)$ is fixed. If $\eta_i = 0$ then η_i is updated to the value of η_j. All other components of η stay unchanged.

(iii) *Exclusion process:* In each step, a vertex $i \in V$ and a neighbouring vertex $j \in V$ are chosen uniformly at random. If $\eta_i = 1$ and $\eta_j = 0$ then the two values are exchanged (i.e., the particle moves from i to j). Otherwise nothing happens.

a) Describe the corresponding stochastic processes as random dynamical systems on an adequate probability space.

b) Show that the processes are Markov chains and identify the transition kernels.

c) Compute the generators $\mathcal{L}f$ for arbitrary functions $f : S \to \mathbb{R}$.

d) Determine invariant probability measures for the three processes. Are these unique?

2. (Markov properties). Let $I = \mathbb{R}_+$ or $I = \mathbb{Z}_+$, and suppose that $(X_t)_{t\in I}$ is a stochastic process with state space $(S_\Delta, \mathcal{B}_\Delta)$ defined on a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$. Show that the following statements are equivalent:

(i) $(X_t, \mathbb{P})$ is a Markov process with initial distribution ν and transition function $(p_{s,t})$.

(ii) For any $n \in \mathbb{Z}_+$ and $0 = t_0 \le t_1 \le \ldots \le t_n$,

$$(X_{t_0}, X_{t_1}, \ldots, X_{t_n}) \sim \nu \otimes p_{t_0,t_1} \otimes p_{t_1,t_2} \otimes \cdots \otimes p_{t_{n-1},t_n} \qquad \text{w.r.t. } \mathbb{P}.$$

(iii) $(X_t)_{t\in I} \sim \mathbb{P}_\nu$.

(iv) For any $s \in I$, $\mathbb{P}^{(s)}_{X_s}$ is a version of the conditional distribution of $(X_t)_{t \geq s}$ given $\mathcal{F}^X_s$, i.e., for any product measurable function $F : S^I_\Delta \to \mathbb{R}_+$,

$$\mathbb{E}[F((X_t)_{t \geq s}) | \mathcal{F}^X_s] \;=\; \mathbb{E}^{(s)}_{X_s}[F] \quad \mathbb{P}\text{-a.s.}$$

Here $\mathbb{P}_\nu$ and $\mathbb{P}^{(s)}_x$ denote the canonical measures on S^I_Δ that correspond to the initial distributions ν, δ_x and the transition functions $(p_{r,t})_{0 \leq r \leq t}$, $(p_{s+r,s+t})_{0 \leq r \leq t}$, respectively.