Institut für Angewandte Mathematik
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“Markov Processes”, Problem Sheet 13

Please hand in your solutions by Friday, January 30, 13.00.

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1. (Exclusion process). Suppose that (V,E) is a regular graph. The *simple exclusion* or *interchange process* is the Markov process with state space $S=\{0,1\}^V$ and transitions $\xi\mapsto\xi^{x,y}$ with rate 1 for every edge $\{x,y\}\in E$. Here $\xi^{x,y}$ denotes the configuration obtained from ξ by exchanging the values at x and y.

a) Show that if V is finite then the number of particles is conserved, and every exchangeable probability measure is invariant. Here a measure μ on S is called *exchangeable* iff $\mu(\xi^{x,y})=\mu(\xi)$ for all $\xi\in S$ and $\{x,y\}\in E$.

b) Show that for every probability measure ν on $\{0,1\}$, the product measure

$$\mu \;=\; \bigotimes_{x\in\mathbb{Z}^d}\nu$$

is invariant for the simple exclusion process on $\{0,1\}^{\mathbb{Z}^d}$.

2. (Images of Markov processes). Suppose that $X_t=f(Y_t)$ where $((Y_t)_{t\geq0},(\mathbb{P}_x)_{x\in S})$ is a time homogeneous Markov process with measurable state space $(S,\mathcal{B})$ and transition function $(p_t)_{t\geq0}$, and f is a measurable map from S to a countable state space T.

a) Derive a necessary and sufficient condition on f ensuring that for all $x\in S$, $((X_t)_{t\geq0},\mathbb{P}_x)$ is again a Markov process, and identify the transition function.

b) The *mean-field contact process* with parameters $b,d\in[0,\infty)$ is the Markov process $(\eta_t)_{t\geq0}$ with state space $\{0,1\}^n$ and transition rates

$$c_1(x,\eta) \;=\; b\cdot N_1(\eta)/n\,, \qquad c_0(x,\eta) \;=\; d\,.$$

Show that $X_t=N_1(\eta_t)$ is a birth-death process and compute the transition rates.

c) How are the generators of the birth-death process and the mean-field voter model related to each other?