Institut für Angewandte Mathematik
Winter semester 2025/26
Andreas Eberle, Francis Lörler

"Markov Processes", Problem Sheet 11

Please hand in your solutions by Friday, January 16, 13.00.

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1. (Sticky Brownian motion). Consider Brownian motion on $[0,1]$ with sticky boundaries at both 0 and 1, with respective parameters λ_0 and λ_1.

a) Write down the generator on the Banach space $C([0,1])$ and verify that it is indeed the generator of a Feller process.

b) Show that there exists an invariant probability measure for the process.

c) Determine the invariant probability measure.

2. (Stochastic dominance I). For two probability measures μ and ν on a partially ordered space $(S, \leq)$, we say that μ is *stochastically dominated by* ν ($\mu \preceq \nu$) iff

$$\int f \, \mathrm{d}\mu \leq \int f \, \mathrm{d}\nu \qquad \text{for every non-decreasing function } f \in \mathcal{F}_b(S).$$

Prove that for $S = \mathbb{R}$, the following statements are equivalent:

(i) $\mu \preceq \nu$.

(ii) $F_\mu(c) = \mu((-\infty, c]) \geq F_\nu(c)$ for all $c \in \mathbb{R}$.

(iii) There exist random variables X and Y defined on a joint probability space such that $X \sim \mu$, $Y \sim \nu$, and $X \leq Y$ almost surely.

3. (Stochastic dominance II). Let μ and ν be probability measures on the configuration space $S = \{0,1\}^{\mathbb{Z}^d}$, endowed with the product topology.

a) Prove that $\mu = \nu$ if and only if $\int f \, \mathrm{d}\mu = \int f \, \mathrm{d}\nu$ for every non-decreasing, bounded and continuous function $f : S \to \mathbb{R}$.

b) Conclude that if $\mu \preceq \nu$ and $\nu \preceq \mu$, then $\mu = \nu$.