Institut für Angewandte Mathematik
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"Markov Processes", Problem Sheet 10

Please hand in your solutions by Friday, January 9, 13.00.

We wish you a merry Christmas and a happy new year!

1. (Brownian motion with absorption at 0). Brownian motion with absorption at 0 is the Markov process with state space $S = [0, \infty)$ defined by $X_t = B_{t \wedge T_0}$, where $(B_t, \mathbb{P}_x)$ is a Brownian motion on $\mathbb{R}$ starting at $x \in [0, \infty)$.

a) Show that this process solves the martingale problem for the operator $\mathcal{L} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with domain $\mathcal{A} = \big\{ f \in C_0^2([0,\infty)) : f''(0) = 0 \big\}$.

b) On which Banach space(s) does this process induce a C^0 contraction semigroup?

c) Identify the corresponding generator(s) $(L, \mathrm{Dom}(L))$.

d) Show that $\int_0^\infty \mathcal{L} f \, \mathrm{d}x = 0$ for any $f \in C_0^\infty(0, \infty)$.

e) Determine all invariant probability measures.

2. (Adjoint processes). Let $(p_t)_{t \geq 0}$ be the transition semigroup of a time-homogeneous Markov process with generator $\mathcal{L}$ on a *finite* state space S. Let μ be a probability measure with full support on S.

a) Write down explicitly the adjoint $\mathcal{L}^*$ of $\mathcal{L}$ as an operator in $L^2(S, \mu)$. Prove that $\mathcal{L}^*$ is the generator of a Markov process if and only if μ is invariant w.r.t. $(p_t)_{t \geq 0}$.

b) Show that in this case, the transition semigroup of the Markov process generated by $\mathcal{L}^*$ is $(p_t^*)_{t \geq 0}$.

c) Give a probabilistic interpretation of this process when μ is the initial distribution.

3. (Differential operators as generators). Suppose that the generator of a Feller semigroup on $\mathbb{R}$ satisfies

$$(Lf)(x) \;=\; \sum_{n=0}^{m} a_n(x) \frac{\mathrm{d}^n f}{\mathrm{d}x^n}(x) \qquad \text{for all } f \in C_0^\infty(\mathbb{R}) \tag{1}$$

for some $m \in \mathbb{N}$ and coefficients $a_n \in C(\mathbb{R})$. Show that for every $x \in \mathbb{R}$,

$$a_0(x) \leq 0, \qquad a_2(x) \geq 0, \qquad \text{and} \qquad a_n(x) = 0 \;\text{ for all } n > 2.$$

4. (Semigroups generated by self-adjoint operators on Hilbert spaces). Suppose that E is a Hilbert space with norm $\|f\| = (f,f)^{1/2}$, and L is a densely defined linear operator on E.

a) Define the *adjoint operator* $(L^*, \mathrm{Dom}(L^*))$. What does it mean that L is *self-adjoint*?

b) Show that if L is *self-adjoint* then it generates a C^0 contraction semigroup on E if and only if L is *negative definite*, i.e.

$$(f, Lf) \leq 0 \quad \text{for all } f \in \mathrm{Dom}(L).$$

Remark. In this case, the C^0 semigroup generated by L is given by $P_t = e^{tL}$, where the exponential is defined by spectral theory, see e.g. Reed & Simon: Methods of modern mathematical physics, Vol. I and II.

5. (Approximation of semigroups by resolvents). Suppose that $(P_t)_{t\geq 0}$ is a strongly continuous contraction semigroup on a closed subspace $E \subseteq \mathcal{F}_b(S)$ with resolvent $(G_\alpha)_{\alpha>0}$.

a) Prove that for any $g \in E$, $t > 0$, $n \in \mathbb{N}$ and $x \in S$,

$$\left(\left(\frac{n}{t} G_{\frac{n}{t}}\right)^n g\right)(x) \;=\; \mathbb{E}\left[\left(P_{\frac{E_1+\cdots+E_n}{n}t}\, g\right)(x)\right]$$

where $(E_k)_{k\in\mathbb{N}}$ is a sequence of independent exponentially distributed random variables with parameter 1.

b) Hence conclude that

$$\left(\frac{n}{t} G_{\frac{n}{t}}\right)^n g \to P_t g \quad \text{uniformly as } n \to \infty. \tag{2}$$

c) How could you derive (2) more directly if the state space is finite?

d) Complete the proof of Step 4 in Theorem 4.21 in the lecture notes. *Hint: You may assume without proof that the probability measures on $\mathbb{R}_+$ with density proportional to $r^{n-1}e^{-nr}$ converge weakly to the Dirac measure δ_1 as $n \to \infty$. (Why?)*

6. (Infinitesimal characterisation of invariant measures – A counterexample). Consider the minimal time-homogeneous Markov jump process $(X_t, \mathbb{P}_x)$ with state space $\mathbb{Z}$ and generator $\mathcal{L} = \lambda\,(\pi - I)$, where

$$\lambda(x) \;=\; 1 + x^2 \qquad \text{and} \qquad \pi(x,\cdot) \;=\; \delta_{x+1} \qquad \text{for all } x \in \mathbb{Z}.$$

a) Give an explicit construction of this process.

b) Does the process explode in finite time?

c) Show that the probability measure μ with weights $\mu(x) \propto 1/(1+x^2)$ is infinitesimally invariant, i.e.

$$(\mu\mathcal{L})(y) \;=\; 0 \qquad \text{for all } y \in \mathbb{Z}.$$

d) Show that nevertheless, μ is not an invariant measure for the transition semigroup (p_t) of the process.