

„Markov Processes”, Problem Sheet 1

Please hand in your solutions before 12 noon on Monday, October 20.

1. (Conditional Expectations).

- a) Let X, Y on $(\Omega, \mathcal{A}, \mathbb{P})$ be independent random variables that are Bernoulli distributed with parameter p . We set $Z = 1_{\{X+Y=0\}}$. Compute $\mathbb{E}[X|Z]$ and $\mathbb{E}[Y|Z]$. Are these random variables still independent?
- b) Let X, Y, Z be random variables with values in a measurable space $(S, \mathcal{B})$ such that the couples (X, Z) and (Y, Z) have the same law. Show that, for any measurable $f : S \rightarrow \mathbb{R}_+$,

$$\mathbb{E}[f(X)|Z] = \mathbb{E}[f(Y)|Z] \quad \text{almost surely.}$$
- c) Let $T_1, \dots, T_n$ be i.i.d. real integrable random variables. Set $T = T_1 + \dots + T_n$.

- (i) Show that $\mathbb{E}[T_1|T] = T/n$.
- (ii) Compute $\mathbb{E}[T|T_1]$.

2. (Markov properties).

- a) Let $(X_t)_{t \in \mathbb{R}_+}$ be a Markov process with transition function $(p_{s,t})$ and state space S on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Show that for any $t_0 \in \mathbb{R}_+$, the time-space process $\hat{X}_t = (t_0 + t, X_{t_0+t})$ is a time-homogeneous Markov process with state space $\mathbb{R}_+ \times S$ and transition function

$$\hat{p}_t((s, x), \cdot) = \delta_{s+t} \otimes p_{s, s+t}(x, \cdot).$$

- b) Let $(X_t)_{t \in \mathbb{R}_+}$ be an arbitrary stochastic process on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Show that the *historical process* $\hat{X}_t = (X_s)_{s \in [0, t]}$ is an $(\mathcal{F}_t^X)$ Markov process.
- c) Let (B_t) be a standard Brownian motion on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and let

$$X_t = \int_0^t B_s ds.$$

Show that (X_t) is not a Markov process, but the two component process (B_t, X_t) is a time-homogeneous Markov process.

- *d) Determine the transition function of this Markov process.

3. (Reflected Random Walks and Metropolis algorithm). Let π be a probability kernel on a measurable space $(S, \mathcal{B})$. A probability measure μ satisfies the *detailed balance condition* w.r.t. π iff for any $A, B \in \mathcal{B}$,

$$(\mu \otimes \pi)(A \times B) = (\mu \otimes \pi)(B \times A).$$

- a) Prove that a probability measure μ which satisfies detailed balance w.r.t. π is also invariant w.r.t. π , i.e., $(\mu\pi)(B) = \int_S \mu(dx) \pi(x, B) = \mu(B)$ for any $B \in \mathcal{B}$.
- b) Let $S \subset \mathbb{R}^d$ be a bounded measurable set. We define a Markov chain $(X_n)_{n \in \mathbb{Z}_+}$ by

$$X_0 = x_0 \in S, \quad X_{n+1} = X_n + W_{n+1} \cdot 1_S(X_n + W_{n+1}),$$

where $W_i : \Omega \rightarrow \mathbb{R}^d$ are i.i.d. random variables. Suppose that the law of W_1 is absolutely continuous with a strictly positive density satisfying $f(x) = f(-x)$. Prove that the uniform distribution on S is invariant w.r.t. the transition kernel of the Markov chain (X_n) .

- c) Let $\mu(dx) = \mu(x) dx$ be a probability measure with strictly positive density on $\mathbb{R}^d$. Show that the process (X_n) , defined by the following algorithm, is a time-homogeneous Markov chain, identify its transition kernel π , and show that μ is invariant for π .

Random walk Metropolis algorithm

- 1.) Set $n := 0$ and choose some arbitrary point $X_0 \in \mathbb{R}^d$.
- 2.) Set $Y_{n+1} := X_n + W_{n+1}$, and draw independently $U_{n+1} \sim \text{Unif}(0, 1)$.
- 3.) If $U_{n+1} < \mu(Y_{n+1})/\mu(X_n)$ then set $X_{n+1} := Y_{n+1}$, else set $X_{n+1} := X_n$.
- 4.) Set $n := n + 1$ and go to Step 2.