Institut für Angewandte Mathematik
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„Markov Processes”, Problem Sheet 0

The exercises on this sheet are optional material in case you are not familiar with general conditional expectations.
See also Williams: Probability with Martingales.

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1. (Revision of conditional expectations 1). Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $\mathcal{F} \subseteq \mathcal{A}$ a σ-algebra, and $X : \Omega \to \mathbb{R}_+$ a non-negative random variable.

a) Define the conditional expectation $\mathbb{E}[X|\mathcal{F}]$.

b) Suppose that there exists a decomposition of Ω into disjoint sets $A_1, \ldots A_n$ such that $\mathcal{F} = \sigma(\{A_1, \ldots, A_n\})$. Show that

$$\mathbb{E}[X|\mathcal{F}] \ = \ \sum\nolimits_{i:\mathbb{P}[A_i]>0} \mathbb{E}[X|A_i]\, 1_{A_i}$$

is a version of the conditional expectation of X given $\mathcal{F}$.

2. (Revision of conditional expectations 2). Let $X, Y : \Omega \to \mathbb{R}_+$ be non-negative random variables. Show that $\mathbb{P}$-almost surely, the following identities hold:

a) For $\lambda \in \mathbb{R}$, we have $\mathbb{E}[\lambda X + Y|\mathcal{F}] = \lambda \mathbb{E}[X|\mathcal{F}] + \mathbb{E}[Y|\mathcal{F}]$.

b) $\mathbb{E}[\mathbb{E}[X|\mathcal{F}]] = \mathbb{E}[X]$ and $|\mathbb{E}[X|\mathcal{F}]| \leq \mathbb{E}[|X|\,|\mathcal{F}]$.

c) If $\sigma(X)$ is independent of $\mathcal{F}$, then $\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X]$.

d) Let $(S, \mathcal{S})$ and $(T, \mathcal{T})$ be measurable spaces. If $Y : \Omega \to S$ is $\mathcal{F}$-measurable, $X : \Omega \to T$ is independent of $\mathcal{F}$, and $f : S \times T \to [0, \infty)$ is product-measurable, then

$$\mathbb{E}[f(Y, X)|\mathcal{F}](\omega) = \mathbb{E}[f(Y(\omega), X)] \qquad \text{for } \mathbb{P}\text{-almost every } \omega \in \Omega.$$

3. (Revision of conditional expectations 3). Let X, Y, Z be random variables on a joint probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We define

$$\mathbb{E}[X|Y] := \mathbb{E}[X|\sigma(Y)].$$

Show the following statements:

a) If $X, Y \in \mathcal{L}^1$ are independent and identically distributed, then $\mathbb{P}$-almost surely

$$\mathbb{E}[X|X+Y] = \frac{1}{2}(X+Y).$$

b) If Z is independent of the pair (X, Y), then $\mathbb{P}$-almost surely

$$\mathbb{E}[X|Y, Z] = \mathbb{E}[X|Y].$$

Is this statement still true if we only assume that X and Z are independent?