Institut für Angewandte Mathematik
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„Markov Processes", Problem Sheet 9

Please hand in your solutions before 12:15 noon on Monday, December 12.

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1. (Differential operators as generators). Suppose that the generator of a Feller semigroup on $\mathbb{R}$ satisfies

$$(Lf)(x) \;=\; \sum_{n=0}^{m} a_n(x)\frac{d^n f}{dx^n}(x) \qquad \text{for all } f \in C_0^\infty(\mathbb{R}) \tag{1}$$

for some $m \in \mathbb{N}$ and coefficients $a_n \in C(\mathbb{R})$. Show that for every $x \in \mathbb{R}$,

$$a_0(x) \le 0, \qquad a_2(x) \ge 0, \qquad \text{and} \qquad a_n(x) = 0 \;\text{ for all } n > 2.$$

2. (Sticky Brownian motion). Consider Brownian motion on $[0,1]$ with sticky boundaries at both 0 and 1, with respective parameters λ_0 and λ_1.

a) Write down the generator on the Banach space $C([0,1])$ and verify that it is indeed the generator of a Feller process.

b) Show that there exists an invariant probability measure for the process.

c) Determine the invariant probability measure.

3. (Infinitesimal characterization of invariant measures - A counterexample). Consider the minimal time-homogeneous Markov jump process $(X_t, \mathbb{P}_x)$ with state space $\mathbb{Z}$ and generator $\mathcal{L} = \lambda\,(\pi - I)$, where

$$\lambda(x) \;=\; 1 + x^2 \qquad \text{and} \qquad \pi(x,\cdot) \;=\; \delta_{x+1} \qquad \text{for all } x \in \mathbb{Z}.$$

a) Give an explicit construction of this process.

b) Does the process explode in finite time?

c) Show that the probability measure μ with weights $\mu(x) \propto 1/(1+x^2)$ is infinitesimally invariant, i.e.

$$(\mu\mathcal{L})(y) \;=\; 0 \qquad \text{for all } y \in \mathbb{Z}.$$

d) Show that nevertheless, μ is not an invariant measure for the transition semigroup (p_t) of the process.