Institut für Angewandte Mathematik
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„Markov Processes", Problem Sheet 8

Please hand in your solutions before 12:15 noon on Monday, December 5.

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1. (Brownian motion on $\mathbb{R}/\mathbb{Z}$). A Brownian motion (X_t) on the circle $\mathbb{R}/\mathbb{Z}$ can be obtained by considering a Brownian motion (B_t) on $\mathbb{R}$ modulo the integers, i.e.,

$$X_t \;=\; B_t - \lfloor B_t \rfloor \;\in\; [0,1) \;\cong\; \mathbb{R}/\mathbb{Z}.$$

Prove the following statements:

a) Brownian motion on $\mathbb{R}/\mathbb{Z}$ is a Markov process with transition density w.r.t. the uniform distribution given by

$$p_t(x,y) \;=\; \frac{1}{\sqrt{2\pi t}} \sum_{n\in\mathbb{Z}} e^{-\frac{|x-y-n|^2}{2t}} \qquad \text{for any } t>0 \text{ and } x,y\in[0,1).$$

b) For any initial condition, (X_t) solves the martingale problem for the operator $\mathcal{L}f = f''/2$ defined on $C^\infty(\mathbb{R}/\mathbb{Z})$. *(Note that there is a one-to-one correspondence of smooth functions on $\mathbb{R}/\mathbb{Z}$ and periodic smooth functions on $\mathbb{R}$ with period 1).*

c) The uniform distribution μ is an invariant probability measure for (p_t).

d) The generator $-\mathcal{L}$ has smooth real-valued eigenfunctions e_n, $n\in\mathbb{Z}$, with corresponding eigenvalues $\lambda_n = 2\pi^2 n^2$. Moreover, $p_t e_n = \exp(-\lambda_n t)e_n$ for any $t\geq 0$.

e) For any $f\in\mathcal{L}^2(\mu)$,

$$\left\| p_t f - \int f\,d\mu \right\|_{L^2(\mu)} \;\leq\; e^{-2\pi^2 t}\sqrt{\mathrm{Var}_\mu(f)}.$$

f) Conclude that for the process with initial distribution μ,

$$E\left[\left(\frac{1}{t}\int_0^t f(X_s)\,ds - \int f\,d\mu\right)^2\right] \;\leq\; \frac{1}{\pi^2 t}\mathrm{Var}_\mu(f) \quad \text{for any } t\geq 0 \text{ and } f\in\mathcal{L}^2(\mu).$$

2. (Brownian motion with absorption at 0). Brownian motion with absorption at 0 is the Markov process with state space $S = [0, \infty)$ defined by $X_t = B_{t \wedge T_0}$, where $(B_t, \mathbb{P}_x)$ is a Brownian motion on $\mathbb{R}$.

a) Show that this process solves the martingale problem for the operator $\mathcal{L} = \frac{1}{2}\frac{d^2}{dx^2}$ with domain $\mathcal{A} = \{ f \in C_0^2([0,\infty)) : f''(0) = 0 \}$.

b) On which Banach space(s) does this process induce a C^0 contraction semigroup?

c) Identify the corresponding generator $(L, \mathrm{Dom}(L))$.

d) Show that $\int_0^\infty \mathcal{L} f \, dx = 0$ for any $f \in C_0^\infty(0, \infty)$.

e) Determine all invariant probability measures.

3. (Approximation of semigroups by resolvents). Suppose that $(P_t)_{t \geq 0}$ is a strongly continuous contraction semigroup on a closed subspace $E \subseteq \mathcal{F}_b(S)$ with resolvent $(G_\alpha)_{\alpha > 0}$.

a) Prove that for any $g \in E$, $t > 0$, $n \in \mathbb{N}$ and $x \in S$,

$$\left(\left(\frac{n}{t} G_{\frac{n}{t}} \right)^n g \right)(x) \;=\; \mathbb{E}\left[\left(P_{\frac{E_1 + \cdots + E_n}{n} t}\, g \right)(x) \right]$$

where $(E_k)_{k \in \mathbb{N}}$ is a sequence of independent exponentially distributed random variables with parameter 1.

b) Hence conclude that

$$\left(\frac{n}{t} G_{\frac{n}{t}} \right)^n g \to P_t g \quad \text{uniformly as } n \to \infty. \tag{1}$$

c) How could you derive (1) more directly if the state space is finite?

d) Complete the proof of Step 4 in Theorem 4.21 in the lecture notes. *Hint: You may assume without proof that the probability measures on $\mathbb{R}_+$ with density proportional to $r^{n-1} e^{-nr}$ converge weakly to the Dirac measure δ_1 as $n \to \infty$. (Why?)*