Institut für Angewandte Mathematik
Wintersemester 2022/23
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„Markov Processes“, Problem Sheet 7

Please hand in your solutions before 12:15 noon on Monday, November 28.

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1. (Semigroups, resolvents and generators).

a) Suppose that $(P_t)_{t\geq 0}$ is a strongly continuous contraction semigroup on a Banach space E with generator $(L, \mathrm{Dom}(L))$, and $(G_\alpha)_{\alpha>0}$ is the corresponding strongly continuous contraction resolvent. Show that for any $f \in \mathrm{Dom}(L)$,

$$G_\alpha(\alpha I - L)f \;=\; f.$$

b) Now suppose that conversely, we are given a densely defined linear operator $(L, \mathrm{Dom}(L))$ on E. State the conditions in the Hille-Yosida Theorem, and verify that under these conditions, $G_\alpha := (\alpha I - L)^{-1}$ is a strongly continuous contraction resolvent.

2. (Strong continuity of transition semigroups of Markov processes on $\mathbf{L^p}$ spaces). Suppose that $(p_t)_{t\geq 0}$ is the transition function of a *right-continuous*, time-homogeneous Markov process $((X_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in S})$, and $\mu \in \mathcal{M}_+(S)$ is a sub-invariant measure.

a) Show that for every $f \in C_b(S)$ and $x \in S$,

$$(p_t f)(x) \to f(x) \qquad \text{as } t \downarrow 0.$$

b) Now let f be a non-negative function in $C_b(S) \cap \mathcal{L}^1(S,\mu)$ and $p \in [1,\infty)$. Show that as $t \downarrow 0$,

$$\int |p_t f - f| d\mu \;\to\; 0\,, \qquad \text{and hence} \qquad p_t f \to f \text{ in } L^p(S,\mu).$$

Hint: You may use that $|x| = x + 2x^-$.

c) Conclude that (p_t) induces a strongly continuous contraction semigroup of linear operators on $L^p(S,\mu)$ for every $p \in [1,\infty)$.

3. (Ornstein-Uhlenbeck process). The transition semigroup of the Ornstein-Uhlenbeck process on $\mathbb{R}$ is given by

$$(p_t f)(x) \;=\; (2\pi)^{-1/2} \int f\left(e^{-t}x + \sqrt{1-e^{-2t}}\, y\right) e^{-y^2/2}\, dy \qquad \text{for } f \in \mathcal{F}_b(\mathbb{R}).$$

a) Show that the standard normal distribution γ is invariant.

b) Denote by C^2_{pol} the space of twice continuously differentiable functions on $\mathbb{R}$ such that f, f' and f'' grow at most polynomially at infinity. Let L denote the generator on $L^2(\mathbb{R}, \gamma)$. Show that $C^2_{\text{pol}} \subset \text{Dom}(L)$ and

$$(Lf)(x) \;=\; f''(x) - xf'(x) \qquad \text{for any } f \in C^2_{\text{pol}}.$$

c) Show that p_t preserves polynomials. Hence conclude that C^2_{pol} is a core for the generator.