Institut für Angewandte Mathematik
Wintersemester 2022/23
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„Markov Processes", Problem Sheet 6

Please hand in your solutions before 12:15 noon on Monday, November 21.

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1. (Strongly continuous semigroups and resolvents).

a) State the defining properties of a strongly continuous contraction semigroup and a strongly continuous contraction resolvent on a Banach space E.

b) Prove that if (P_t) is a C_0 contraction semigroup then $G_\alpha f = \int_0^\infty e^{-\alpha t} P_t f \, dt$ defines a C_0 contraction resolvent.

2. (Uniform motion to the right). Consider the deterministic Markov process $(X_t, \mathbb{P}_x)$ on $\mathbb{R}$ given by $X_t = x + t$ $\mathbb{P}_x$-almost surely.

a) Show that the transition semigroup $(P_t)_{t \geq 0}$ is strongly continuous both on $\hat{C}(\mathbb{R})$ and on $L^2(\mathbb{R}, dx)$.

b) Prove that the generator on $\hat{C}(\mathbb{R})$ is given by

$$Lf \;=\; f', \qquad \mathrm{Dom}(L) \;=\; \{f \in C^1(\mathbb{R}) : f, f' \in \hat{C}(\mathbb{R})\}.$$

c) Show that the generator on $L^2(\mathbb{R}, dx)$ is given by

$$Lf \;=\; f', \qquad \mathrm{Dom}(L) \;=\; H^{1,2}(\mathbb{R}, dx).$$

3. (Brownian motion killed at 0). Let $X_t = B_t$ for $t < T$ and $X_t = \Delta$ for $t \geq T$, where $(B_t)_{t \geq 0}$ is a one-dimensional Brownian motion and $T = \inf\{t \geq 0 : B_t = 0\}$.

a) Show that $(X_t)_{t \geq 0}$ is a Markov process on the extended state space $(0, \infty) \dot{\cup} \{\Delta\}$ with transition kernel satisfying $p_t^{\mathrm{Dir}}(x, B) = \int_B p_t^{\mathrm{Dir}}(x, y) \, dy$ for all $x \in (0, \infty)$ and $B \in \mathcal{B}((0, \infty))$, where

$$p_t^{\mathrm{Dir}}(x, y) \;=\; \frac{1}{\sqrt{2\pi t}} \left(\exp\left(-\frac{(y-x)^2}{2t} \right) - \exp\left(-\frac{(y+x)^2}{2t} \right) \right) \quad \text{for } x, y \in (0, \infty).$$

b) We extend functions $f : (0, \infty) \to \mathbb{R}$ to the extended state space $(0, \infty) \dot{\cup} \{\Delta\}$ by setting $f(\Delta) := 0$, Prove that in this sense, $(X_t, \mathbb{P})$ solves the martingale problem for the operator $\mathcal{L} f = \frac{1}{2} f''$ with domain

$$\mathcal{A} \;=\; \{f \in C_b^2([0, \infty)) : f(0) = 0\}.$$

4. (Tightness). Prove the following three statements.

a) A sequence of probability measures on the line is tight if and only if, for the corresponding distribution functions, we have $\lim_{x\to\infty} F_n(x) = 1$ and $\lim_{x\to-\infty} F_n(x) = 0$ uniformly in n.

b) A sequence of normal distributions on the line is tight if and only if the means and the variances are bounded (a normal distribution with variance 0 being a point mass).

c) A sequence of distributions of random variables X_n is tight if (X_n) is uniformly integrable.

Reminder: A sequence of random variables X_n is uniformly integrable if

$$\sup_{n\in\mathbb{N}} E[|X_n|; |X_n| \geq c] \to 0 \text{ as } c \to \infty.$$