Institut für Angewandte Mathematik
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„Markov Processes", Problem Sheet 4

Please hand in your solutions before 12:15 noon on Monday, November 7.

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1. (Recurrence on discrete state spaces). Let $(X_n, \mathbb{P}_x)$ be an irreducible time-homogeneous Markov chain with countable state space S.

a) Let $x, y \in S$. Show that if x is recurrent then

$$P_x[X_n = y \text{ infinitely often }] = 1,$$

and y is also recurrent.
Hint: Let $T^{(0)} := 0$ and let $0 < T^{(1)} < T^{(2)} < \dots$ denote the sequence of return times to the state x. Show that under $\mathbb{P}_x$, the events $A_k := \{X_n = y$ for some $n \in (T^{(k-1)}, T^{(k)})\}$, $k \in \mathbb{N}$, are independent with equal probability $p > 0$. Hence conclude that $\mathbb{P}_x$-almost surely, infinitely many of these events occur.

b) Prove that the following statements are all equivalent:

(i) There exists a finite recurrent set $A \subseteq S$.

(ii) There exists $x \in S$ such that the set $\{x\}$ is recurrent.

(iii) For any $x \in S$, the set $\{x\}$ is recurrent.

(iv) For any $x, y \in S$,
$$\mathbb{P}_x[X_n = y \text{ infinitely often }] = 1.$$

c) Show using Lyapunov functions that the simple random walk on $\mathbb{Z}^2$ is recurrent. *Hint: Consider for example the functions $V(x) = (\log(1 + |x|^2))^\alpha$ for $\alpha > 0$.*

2. (Positive recurrence, invariance and ergodicity on discrete state spaces). Let $(X_n, \mathbb{P}_x)$ be a time-homogeneous Markov chain with countable state space S, and suppose that $x \in S$ is positive recurrent.

a) Show that the measure

$$\nu(B) \;=\; \mathbb{E}_x\left[\sum_{i=0}^{T_x^+ - 1} 1_B(X_i)\right], \qquad B \subseteq S,$$

is invariant, and $\mu(B) = \nu(B)/\mathbb{E}_x[T_x^+]$ is an invariant probability measure.

b) Let $T^{(0)} := 0$ and let $0 < T^{(1)} < T^{(2)} < \cdots$ denote the sequence of return times to the state x. Show that for every $B \subseteq S$, the random variables

$$V_k(B) \;:=\; \sum_{i=T^{(k-1)}}^{T^{(k)}-1} 1_B(X_i), \qquad k \in \mathbb{N},$$

are i.i.d. under $\mathbb{P}_x$ with $\mathbb{E}_x[V_k(B)] = \nu(B)$. Conclude that for all $k \in \mathbb{N}$,

$$\mathbb{E}_x\left[T^{(k)}\right] \;=\; k \cdot \mathbb{E}_x\left[T_x^+\right].$$

c) Prove that as $n \to \infty$,

$$\frac{1}{n}\sum_{i=0}^{n-1} 1_B(X_i) \;\longrightarrow\; \mu(B) \qquad \mathbb{P}_x\text{-almost surely for every } B \subseteq S.$$

3. (Weak convergence of probability measures). Let μ_n $(n \in \mathbb{N})$ and μ be probability measures on the Borel σ-algebra over a metric space S. Prove that the following statements are equivalent:

(i) The sequence $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to μ.

(ii) For any bounded and Lipschitz continuous function $f : S \to \mathbb{R}$,

$$\int f\, d\mu_n \text{ converges to } \int f\, d\mu.$$

(iii) For any closed set $A \subseteq S$,

$$\limsup \mu_n[A] \;\le\; \mu[A].$$

(vi) For any open set $O \subseteq S$,

$$\liminf \mu_n[O] \;\ge\; \mu[O].$$

(v) For any $B \in \mathcal{B}(S)$ such that $\mu[\partial B] = 0$,

$$\lim \mu_n[B] \;=\; \mu[B].$$