Institut für Angewandte Mathematik
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„Markov Processes", Problem Sheet 3

Please hand in your solutions before 12:15 noon on Monday, October 31.

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1. (Martingales and boundary value problems). Let $(X_n, \mathbb{P}_x)$ be a canonical time-homogeneous Markov chain with measurable state space $(S, \mathcal{B})$ and generator $\mathcal{L}$, and let

$$T = \inf\{n \geq 0 : X_n \notin D\}$$

denote the first exit time from a subset $D \in \mathcal{B}$.

a) Suppose that $v : S \to \mathbb{R}$ is a non-negative measurable function satisfying

$$\mathcal{L}v \leq -c \quad \text{in } D$$

for a non-negative function $c : D \to \mathbb{R}$. Prove that for every $x \in S$, the process

$$M_n = v(X_{n\wedge T}) + \sum_{i<n\wedge T} c(X_i) \tag{1}$$

is a non-negative supermartingale w.r.t. $\mathbb{P}_x$.

b) The second part of this exercise gives a direct proof of the uniqueness part in the Feynman-Kac formula for Markov chains. Let $w : S \to \mathbb{R}$ be a non-negative measurable function. Determine for which functions v the process

$$M_n = e^{-\sum_{k=0}^{n-1} w(X_i)} v(X_n)$$

is a martingale. Now suppose that for all $x \in S$, $T < \infty$ holds $\mathbb{P}_x$-almost surely, and let v be a bounded solution to the boundary value problem

$$\begin{aligned} (\mathcal{L}v)(x) &= (e^{w(x)} - 1)v(x) && \text{for all } x \in D, \\ v(x) &= f(x) && \text{for all } x \in S \setminus D. \end{aligned} \tag{2}$$

Show that

$$v(x) = \mathbb{E}_x\left[e^{-\sum_{k=0}^{T-1} w(X_k)} f(X_T)\right].$$

2. (Random walks on $\mathbb{Z}$). Suppose that $(X_n, \mathbb{P}_x)$ is a time-homogeneous Markov chain with state space $\mathbb{Z}$ and transition matrix π given by

$$\pi(x, x+1) = p, \quad \pi(x, x) = r, \quad \pi(x, x-1) = q,$$

where $p + q + r = 1$, $p > 0$, $q > 0$ and $r \geq 0$. Fix $a, b \in \mathbb{Z}$ with $a < b - 1$ and let

$$T \;=\; \inf\{n \geq 0 : X_n \notin (a, b)\}.$$

a) Prove that for every function $g : \{a+1, a+2, \ldots, b-1\} \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$, the system

$$\begin{aligned} (\mathcal{L}u)(x) &= -g(x) \quad \text{for } a < x < b, \\ u(a) &= \alpha, \quad u(b) \;=\; \beta, \end{aligned} \tag{3}$$

has a unique solution.

b) Conclude that $\mathbb{E}_x[T] < \infty$ for all x.

c) How can the mean exit time be computed explicitly? Carry out the computation in the case where $p = q$ and $a = -b$.

Hint: Write down a system of equations for the "first derivative" $v(x) := u(x) - u(x-1)$.

3. (Strong Markov property in continuous time). Suppose that $(X_t, \mathbb{P}_x)$ is a time homogeneous $(\mathcal{F}_t)$ Markov process in continuous time with state space $\mathbb{R}^d$ and transition semigroup (p_t).

a) Let T be an $(\mathcal{F}_t)$ stopping time taking only the discrete values $t_i = ih$, $i \in \mathbb{Z}_+$, for some fixed $h \in (0, \infty)$. Prove that for any initial value $x \in \mathbb{R}^d$ and any non-negative measurable function $F : (\mathbb{R}^d)^{[0,\infty)} \to \mathbb{R}$,

$$E_x\left[F(X_{T+\bullet})|\mathcal{F}_T\right] \;=\; E_{X_T}[F(X)] \qquad P_x\text{-almost surely.} \tag{4}$$

b) The transition semigroup (p_t) is called *Feller* iff for every $t \geq 0$ and every bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$, $x \mapsto (p_t f)(x)$ is continuous. Prove that if $t \mapsto X_t(\omega)$ is right continuous for all ω and (p_t) is a Feller semigroup, then for *every* $(\mathcal{F}_t)$ stopping time $T : \Omega \to [0, \infty)$,

$$\mathbb{E}_x\left[f(X_{T+t})|\mathcal{F}_T\right] \;=\; \mathbb{E}_{X_T}[f(X_t)] \tag{5}$$

holds P_x-almost surely for all $t \geq 0$ and all $f \in C_b(\mathbb{R}^d)$.

c) Conclude that, under the assumptions of b), the strong Markov property (4) holds for every $(\mathcal{F}_t)$ stopping time $T : \Omega \to [0, \infty)$.