Institut für Angewandte Mathematik
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„Markov Processes", Problem Sheet 2

Please hand in your solutions before 12:15 noon on Monday, October 24.

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1. (Markov properties). Let $I = \mathbb{R}_+$ or $I = \mathbb{Z}_+$, and suppose that $(X_t)_{t\in I}$ is a stochastic process with state space $(S_\Delta, \mathcal{B}_\Delta)$ defined on a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$. Show that the following statements are equivalent:

(i) $(X_t, \mathbb{P})$ is a Markov process with initial distribution ν and transition function $(p_{s,t})$.

(ii) For any $n \in \mathbb{Z}_+$ and $0 = t_0 \leq t_1 \leq \ldots \leq t_n$,

$$(X_{t_0}, X_{t_1}, \ldots, X_{t_n}) \sim \nu \otimes p_{t_0,t_1} \otimes p_{t_1,t_2} \otimes \cdots \otimes p_{t_{n-1},t_n} \qquad \text{w.r.t. } \mathbb{P}.$$

(iii) $(X_t)_{t\in I} \sim \mathbb{P}_\nu$.

(iv) For any $s \in I$, $\mathbb{P}^{(s)}_{X_s}$ is a version of the conditional distribution of $(X_t)_{t\geq s}$ given $\mathcal{F}^X_s$, i.e., for any product measurable function $F : S^I_\Delta \to \mathbb{R}_+$,

$$\mathbb{E}[F((X_t)_{t\geq s})|\mathcal{F}^X_s] \;=\; \mathbb{E}^{(s)}_{X_s}[F] \quad \mathbb{P}\text{-a.s.}$$

Here $\mathbb{P}_\nu$ and $\mathbb{P}^{(s)}_x$ denote the canonical measures on S^I_Δ that correspond to the initial distributions ν, δ_x and the transition functions $(p_{r,t})_{0\leq r\leq t}$, $(p_{s+r,s+t})_{0\leq r\leq t}$, respectively.

2. (Reduction to the time-homogeneous case). Suppose that $((X_t)_{t\in\mathbb{Z}_+}, \mathbb{P})$ is a Markov chain with state space $(S, \mathcal{B})$ and one step transition kernels π_t, $t \in \mathbb{N}$.

a) Determine the transition kernel and the generator of the time-space process (t, X_t).

b) Conclude that for any function $f \in \mathcal{F}_b(\mathbb{Z}_+ \times S)$, the process

$$M^{[f]}_t \;=\; f(t, X_t) - \sum_{k=0}^{t-1} \mathcal{L}_k(f(k+1,\cdot))(X_k) - \sum_{k=0}^{t-1}(f(k+1, X_k) - f(k, X_k))$$

is a martingale, where $(\mathcal{L}_t)$ are the generators of (X_t).

c) What would be a corresponding statement in continuous time (without proof) ?

3. (Brownian motion reflected at 0). Let $(B_t)_{t\geq 0}$ be a one-dimensional Brownian motion defined on a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$.

a) Show that $X_t = |B_t|$ is a Markov process with transition density

$$p_t^{\text{refl}}(x, y) \;=\; \frac{1}{\sqrt{2\pi t}} \left(\exp\left(-\frac{(y-x)^2}{2t} \right) + \exp\left(-\frac{(y+x)^2}{2t} \right) \right).$$

b) Prove that $(X_t, \mathbb{P})$ solves the martingale problem for the operator $\mathcal{L}f = \frac{1}{2}f''$ with domain

$$\mathcal{A} \;=\; \{ f \in C_b^2([0,\infty)) : f'(0) = 0 \}.$$

Hint: Note that functions in $\mathcal{A}$ can be extended to symmetric functions in $C_b^2(\mathbb{R})$.

c) Construct another solution to the martingale problem for $\mathcal{L}$ with domain $C_0^\infty(0,\infty)$. Does it also solve the martingale problem in b) ?

4. (Strong Markov property and Harris recurrence). Let $(X_n, \mathbb{P}_x)$ be a time homogeneous $(\mathcal{F}_n)$ Markov chain on the state space $(S, \mathcal{B})$ with transition kernel $\pi(x, dy)$.

a) Show that if T is a finite $(\mathcal{F}_n)$ stopping time, then conditionally given $\mathcal{F}_T$, the process $\hat{X}_n := X_{T+n}$ is a Markov chain with transition kernel π starting in X_T.

b) Conclude that a set $A \in \mathcal{B}$ is *Harris recurrent*, i.e.,

$$\mathbb{P}_x[X_n \in A \text{ for some } n \geq 1] = 1 \quad \text{for any } x \in A,$$

if and only if

$$\mathbb{P}_x[X_n \in A \text{ infinitely often}] = 1 \quad \text{for any } x \in A.$$