Institut für Angewandte Mathematik
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„Markov Processes", Problem Sheet 13

Please hand in your solutions before 12:15 noon on Monday, January 23.

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1. (Ergodicity and decay of correlations). Suppose that $(X_t)_{t\in[0,\infty)}$ on $(\Omega, \mathcal{A}, P)$ is a canonical stationary stochastic process.

a) Prove that the following properties are equivalent:

(i) P is ergodic.

(ii) $\mathrm{Var}\left(\frac{1}{t}\int_0^t F\circ\theta_s\,ds\right) \to 0$ as $t\uparrow\infty$ for any $F \in \mathcal{L}^2(\Omega, \mathcal{A}, P)$.

(iii) $\frac{1}{t}\int_0^t \mathrm{Cov}\,(F\circ\theta_s, G)\,ds \to 0$ as $t\uparrow\infty$ for any $F, G \in \mathcal{L}^2(\Omega, \mathcal{A}, P)$.

(iv) $\frac{1}{t}\int_0^t \mathrm{Cov}\,(F\circ\theta_s, F)\,ds \to 0$ as $t\uparrow\infty$ for any $F \in \mathcal{L}^2(\Omega, \mathcal{A}, P)$.

b) The process (X_t) is said to be **mixing** iff

$$\lim_{t\to\infty} \mathrm{Cov}\,(F\circ\theta_t\,,\,G) \;=\; 0 \qquad \text{for any } F, G \in \mathcal{L}^2(\Omega, \mathcal{A}, P).$$

Prove that:

(i) If $(X_t)_{t\geq 0}$ is mixing then it is ergodic.

(ii) If the tail σ-algebra $\mathcal{T} = \bigcap_{t\geq 0}\sigma(X_s : s\geq t)$ is trivial in the sense that $P[A] \in \{0,1\}$ for any $A\in\mathcal{T}$, then $(X_t)_{t\geq 0}$ is mixing (and hence ergodic).

2. (Ergodicity and irreducibility for Markov processes in continuous time). We consider a canonical Markov process $((X_t)_{t\geq 0}, (P_x)_{x\in S})$ with state space $(S, \mathcal{B})$ and transition semigroup $(p_t)_{t\geq 0}$.

a) Show that for $\mu \in \mathcal{P}(S)$, the following three conditions are equivalent:

(i) $P_\mu \circ \theta_t^{-1} = P_\mu$ for any $t \geq 0$.

(ii) $((X_t)_{t\geq 0}, P_\mu)$ is a stationary process.

(iii) μ is invariant with respect to p_t for all $t\geq 0$.

b) From now on we assume that the process is stationary. Show that for every shift-invariant event A, there exists a set $B\in\mathcal{B}$ such that $p_t 1_B = 1_B$ μ-almost surely for any $t\geq 0$, and

$$1_A = 1_{\bigcup_{n\in\mathbb{N}}\bigcap_{m\geq n}\{X_m\in B\}} = 1_{\{X_0\in B\}} \quad P_\mu\text{-almost surely.}$$

c) Show that the following four conditions are equivalent:

(i) P_μ is ergodic.

(ii) The kernel of the $L^2(\mu)$ generator $L^{(2)}$ contains only equivalence classes of constant functions.

(iii) Every function $h \in \mathcal{L}^2(\mu)$ such that $p_t h = h$ μ-a.s. $\forall t \geq 0$ is almost surely constant.

(iv) Every set $B \in \mathcal{B}$ such that $p_t 1_B = 1_B$ μ-a.s. for any $t \geq 0$ satisfies $\mu(B) \in \{0, 1\}$.