Institut für Angewandte Mathematik
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Andreas Eberle, Stefan Oberdörster

„Markov Processes", Problem Sheet 12

Please hand in your solutions before 12:15 noon on Monday, January 16.

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1. (Ergodicity for stationary processes). Let $(X_n)_{n\in\mathbb{Z}_+}$ on $(\Omega, \mathcal{A}, \mathbb{P})$ be a canonical stationary process with state space S, and let $\mathcal{J} = \{A \in \mathcal{A} : A = \Theta^{-1}(A)\}$.

a) Show that $\mathbb{P}$ is not ergodic if and only if there exists a decomposition $\Omega = A \cup A^c$ into disjoint sets $A, A^c \in \mathcal{A}$ such that $\mathbb{P}[A] > 0$, $\mathbb{P}[A^c] > 0$,

$$\Theta(A) \subseteq A \quad \text{and} \quad \Theta(A^c) \subseteq A^c \,.$$

b) Prove that for $F : \Omega \to \mathbb{R}$ the following two properties are equivalent:

(i) F is $\mathcal{J}$-measurable.

(ii) $F = F \circ \Theta$.

c) Conclude that $\mathbb{P}$ is ergodic if and only if every shift-invariant function $F : \Omega \to \mathbb{R}$ is $\mathbb{P}$-almost surely constant.

2. (Ergodicity for stationary Markov chains). Let $((X_n)_{n\in\mathbb{Z}_+}, (\mathbb{P}_x)_{x\in S})$ be a canonical time-homogeneous Markov chain, and let μ be an invariant probability measure for the transition kernel π.

a) Show that the following properties are all equivalent:

(i) $\mathbb{P}_\mu$ is ergodic.

(ii) $\frac{1}{n}\sum_{i=0}^{n-1} F \circ \Theta^i \;\to\; E_\mu[F]$ $\mathbb{P}_\mu$-almost surely for any $F \in \mathcal{L}^1(\mathbb{P}_\mu)$.

(iii) $\frac{1}{n}\sum_{i=0}^{n-1} f(X_i) \;\to\; \int f\, d\mu$ $\mathbb{P}_\mu$-almost surely for any $f \in \mathcal{L}^1(\mu)$.

(iv) For any $B \in \mathcal{B}$,

$$\frac{1}{n}\sum_{i=0}^{n-1} \pi^i(x, B) \;\to\; \mu(B) \qquad \text{for } \mu\text{-almost every } x.$$

(v) For any $B \in \mathcal{B}$ such that $\mu(B) > 0$,

$$\mathbb{P}_x[T_B < \infty] > 0 \qquad \text{for } \mu\text{-almost every } x.$$

(vi) Any set $B \in \mathcal{B}$ satisfying $\pi(x, B) = 1_B(x)$ for μ-a.e. x has measure $\mu(B) \in \{0, 1\}$.

b) Prove that $\mathbb{P}_\mu$ is ergodic if the support of μ is connected, and the transition kernel π has the *strong Feller property* (i.e., πf is continuous for every $f \in \mathcal{F}_b(S)$).

c) Now suppose that the state space is $\mathbb{R}^d$, and

$$\pi(x, B) \;=\; \int_B \pi(x, y)\, \lambda^d(dy), \qquad B \in \mathcal{B}(\mathbb{R}^d),$$

where the transition density $\pi(x, y)$ is a continuous and strictly positive function such that for every $r \in (0, \infty)$, $\int \sup_{|x| \le r} \pi(x, y)\, \lambda^d(dy) < \infty$. Show that $\mathbb{P}_\mu$ is ergodic.