Institut für Angewandte Mathematik
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Andreas Eberle, Stefan Oberdörster

„Markov Processes", Problem Sheet 11

Please hand in your solutions before 12:15 noon on Monday, January 9.

We wish you a merry Christmas and a happy new year!

1. (Exclusion process and independent particle process).

a) Show that for every probability measure ν on $\{0,1\}$, the product measure

$$\mu \;=\; \bigotimes_{x\in\mathbb{Z}^d} \nu$$

is invariant for the simple exclusion process on $\{0,1\}^{\mathbb{Z}^d}$.

Remark: In the lecture, we have shown a more general result for the simple exclusion process on a finite graph. However, the proof does not carry over to infinite graphs.

c) Suppose that $\left(X_t^k\right)_{t\geq 0}$ $(k \in \mathbb{N})$ are independent continuous time simple random walks on $\mathbb{Z}^d$, and let

$$\eta_t(x) \;=\; \left|\{k \in \mathbb{N} : X_t^k = x\}\right| \,.$$

Assuming that $\eta_t(x)$ is almost surely finite for all $x \in \mathbb{Z}^d$ and $t \geq 0$, show that $(\eta_t)_{t\geq 0}$ is a Markov process with state space $\{0,1,2,\ldots\}^{\mathbb{Z}^d}$, and identify the generator.

2. (Stochastic dominance I). For two probability measures μ and ν on a partially ordered space $(S,\leq)$, we say that μ is *stochastically dominated by* ν ($\mu \preceq \nu$) iff

$$\int f \,\mathrm{d}\mu \leq \int f \,\mathrm{d}\nu \qquad \text{for every non-decreasing function } f \in \mathcal{F}_b(S).$$

Prove that for $S = \mathbb{R}$, the following statements are equivalent:

(i) $\mu \preceq \nu$.

(ii) $F_\mu(c) = \mu((-\infty, c]) \geq F_\nu(c)$ for all $c \in \mathbb{R}$.

(iii) There exist random variables X and Y defined on a joint probability space such that $X \sim \mu$, $Y \sim \nu$, and $X \leq Y$ almost surely.

3. (Stochastic dominance II). Let μ and ν be probability measures on the configuration space $S = \{0,1\}^{\mathbb{Z}^d}$, endowed with the product topology.

a) Prove that $\mu = \nu$ if and only if $\int f \, \mathrm{d}\mu = \int f \, \mathrm{d}\nu$ for every non-decreasing, bounded and continuous function $f : S \to \mathbb{R}$.

b) Conclude that if $\mu \preceq \nu$ and $\nu \preceq \mu$, then $\mu = \nu$.