Institut für Angewandte Mathematik
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„Markov Processes", Problem Sheet 10

Please hand in your solutions before 12:15 noon on Monday, December 19.

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1. (Interacting particle systems on finite graphs).

a) Show that for the voter model on a finite graph, the processes

$$N_i(\eta_t) \;=\; \sum_{x\in V} 1_i(\eta_t(x))$$

are martingales. Conclude that for all i and all initial configurations ξ,

$$\mathbb{P}_\xi\left[\eta_t \equiv i \text{ eventually}\right] \;=\; \frac{N_i(\xi)}{|V|}.$$

b) Prove for the contact process on a finite graph that for all $\xi \in \{0,1\}^V$,

$$\mathbb{P}_\xi\left[\eta_t \equiv 0 \text{ eventually}\right] \;=\; 1.$$

2. (Adjoint processes). Let $(p_t)_{t\geq 0}$, be the transition semigroup of a time-homogeneous Markov process with generator $\mathcal{L}$ on a *finite* state space S. Let μ be a probability measure with full support on S.

a) Write down explicitly the adjoint $\mathcal{L}^*$ of $\mathcal{L}$ as an operator in $L^2(S,\mu)$. Prove that $\mathcal{L}^*$ is the generator of a Markov process if and only if μ is invariant w.r.t. $(p_t)_{t\geq 0}$.

b) Show that in this case, the transition semigroup of the Markov process generated by $\mathcal{L}^*$ is $(p_t^*)_{t\geq 0}$.

c) Give a probabilistic interpretation of this process when μ is the initial distribution.

3. (Semigroups generated by self-adjoint operators on Hilbert spaces). Suppose that E is a Hilbert space with norm $\|f\| = (f,f)^{1/2}$, and L is a densely defined linear operator on E.

a) Define the *adjoint operator* $(L^*, \mathrm{Dom}(L^*))$. What does it mean that L is *self-adjoint*?

b) Show that if L is *self-adjoint* then it generates a C^0 contraction semigroup on E if and only if L is *negative definite*, i.e.

$$(f, Lf) \leq 0 \quad \text{ for all } f \in \mathrm{Dom}(L).$$

Remark. In this case, the C^0 semigroup generated by L is given by $P_t = e^{tL}$, where the exponential is defined by spectral theory, see e.g. Reed & Simon: Methods of modern mathematical physics, Vol. I and II.