Institut für Angewandte Mathematik
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„Markov Processes", Problem Sheet 1

Please hand in your solutions before 12 noon on Monday, October 17.

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1. (Conditional Expectations).

a) Let X, Y on $(\Omega, \mathcal{A}, \mathbb{P})$ be independent random variables that are Bernoulli distributed with parameter p. We set $Z = 1_{\{X+Y=0\}}$. Compute $\mathbb{E}[X|Z]$ and $\mathbb{E}[Y|Z]$. Are these random variables still independent?

b) Let X, Y, Z be random variables with values in a measurable space $(S, \mathcal{B})$ such that the couples (X, Z) and (Y, Z) have the same law. Show that, for any measurable $f : S \to \mathbb{R}_+$,

$$\mathbb{E}[f(X)|Z] \;=\; \mathbb{E}[f(Y)|Z] \quad \text{almost surely.}$$

c) Let $T_1, \dots, T_n$ be i.i.d. real integrable random variables. Set $T = T_1 + \cdots + T_n$.

(i) Show that $\mathbb{E}[T_1|T] = T/n$.

(ii) Compute $\mathbb{E}[T|T_1]$.

2. (Markov properties).

a) Let $(X_t)_{t\in\mathbb{R}_+}$ be a Markov process with transition function $(p_{s,t})$ and state space S on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Show that for any $t_0 \in \mathbb{R}_+$, the time-space process $\hat{X}_t = (t_0 + t, X_{t_0+t})$ is a time-homogeneous Markov process with state space $\mathbb{R}_+ \times S$ and transition function

$$\hat{p}_t\left((s, x), \cdot\right) \;=\; \delta_{s+t} \otimes p_{s,s+t}(x, \cdot).$$

b) Let $(X_t)_{t\in\mathbb{R}_+}$ be an arbitrary stochastic process on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Show that the *historical process* $\hat{X}_t = (X_s)_{s\in[0,t]}$ is an $(\mathcal{F}^X_t)$ Markov process.

c) Let (B_t) be a standard Brownian motion on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and let

$$X_t \;=\; \int_0^t B_s\, ds.$$

Show that (X_t) is not a Markov process, but the two component process (B_t, X_t) is a time-homogeneous Markov process.

*d) Determine the transition function of this Markov process.

3. (Reflected Random Walks and Metropolis algorithm). Let π be a probability kernel on a measurable space $(S, \mathcal{B})$. A probability measure μ satisfies the *detailed balance condition* w.r.t. π iff for any $A, B \in \mathcal{B}$,

$$(\mu \otimes \pi)(A \times B) \;=\; (\mu \otimes \pi)(B \times A).$$

a) Prove that a probability measure μ which satisfies detailed balance w.r.t. π is also invariant w.r.t. π, i.e., $(\mu\pi)(B) = \int_S \mu(dx)\, \pi(x, B) = \mu(B)$ for any $B \in \mathcal{B}$.

b) Let $S \subset \mathbb{R}^d$ be a bounded measurable set. We define a Markov chain $(X_n)_{n \in \mathbb{Z}_+}$ by

$$X_0 \;=\; x_0 \in S, \qquad X_{n+1} \;=\; X_n + W_{n+1} \cdot 1_S(X_n + W_{n+1}),$$

where $W_i : \Omega \to \mathbb{R}^d$ are i.i.d. random variables. Suppose that the law of W_1 is absolutely continuous with a strictly positive density satisfying $f(x) = f(-x)$. Prove that the uniform distribution on S is invariant w.r.t. the transition kernel of the Markov chain (X_n).

c) Let $\mu(dx) = \mu(x)\, dx$ be a probability measure with strictly positive density on $\mathbb{R}^d$. Show that the process (X_n), defined by the following algorithm, is a time-homogeneous Markov chain, identify its transition kernel π, and show that μ is invariant for π.

Random walk Metropolis algorithm
1.) Set $n := 0$ and choose some arbitrary point $X_0 \in \mathbb{R}^d$.
2.) Set $Y_{n+1} := X_n + W_{n+1}$, and draw independently $U_{n+1} \sim \text{Unif}(0, 1)$.
3.) If $U_{n+1} < \mu(Y_{n+1})/\mu(X_n)$ then set $X_{n+1} := Y_{n+1}$, else set $X_{n+1} := X_n$.
4.) Set $n := n + 1$ and go to Step 2.