Institute for Applied Mathematics
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"Markov Processes", Problem Sheet 14.

Exercises to be discussed in tutorials on Wednesday 8.2.

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1. (Càdlàg functions). Let $x : [0, \infty) \to S$ be a right continuous function with left limits taking values in a complete separable metric space (S, d). Show that for any $T \in \mathbb{R}_+$,

$$\omega'_{\delta,T}(x) \;:=\; \inf_{|t_i - t_{i-1}| \geq \delta} \max_i \sup_{s,t \in [t_{i-1}, t_i)} d(x(s), x(t)) \;\longrightarrow\; 0 \qquad \text{as } \delta \downarrow 0.$$

Here the infimum is over all partitions $0 = t_0 < t_1 < \ldots < t_{n-1} < T < t_n$ such that $|t_i - t_{i-1}| \geq \delta$ for any $i \leq n$.

2. (Strongly continuous semigroups and resolvents).

a) State the defining properties of a strongly continuous contraction semigroup and a strongly continuous contraction resolvent on a Banach space E.

b) Prove that if (P_t) is a C_0 contraction semigroup then $G_\alpha f = \int_0^\infty e^{-\alpha t} P_t f \, dt$ defines a C_0 contraction resolvent.

c) Compute the resolvent of Brownian motion on $\hat{C}(\mathbb{R})$ explicitly.

3. (Uniform motion to the right). Consider a deterministic Markov process (X_t, P_x) on $\mathbb{R}$ given by $X_t = x + t$ P_x-almost surely.

a) Show that the transition semigroup $(P_t)_{t \geq 0}$ is strongly continuous both on $\hat{C}(\mathbb{R})$ and on $L^2(\mathbb{R}, dx)$.

b) Prove that the generator on $\hat{C}(\mathbb{R})$ is given by

$$Lf \;=\; f', \qquad \mathrm{Dom}(L) \;=\; \{f \in \hat{C}(\mathbb{R}) : f' \in \hat{C}(\mathbb{R})\}.$$

c) Show that the generator on $L^2(\mathbb{R}, dx)$ is given by

$$Lf \;=\; f', \qquad \mathrm{Dom}(L) \;=\; H^{1,2}(\mathbb{R}, dx).$$

4. (Ornstein-Uhlenbeck process). The transition semigroup of the Ornstein-Uhlenbeck process on $\mathbb{R}$ is given by

$$(p_t f)(x) \;=\; (2\pi)^{-1/2} \int f\left(e^{-t}x + \sqrt{1-e^{-2t}}\, y\right)\, e^{-y^2/2}\, dy \qquad \text{for } f \in \mathcal{F}_b(\mathbb{R}).$$

a) Show that the standard normal distribution γ is invariant.

b) Let L denote the generator on $L^2(\mathbb{R}, \gamma)$. Show that $C^2_{\text{pol}} \subset \mathrm{Dom}(L)$ and

$$(Lf)(x) \;=\; f''(x) - xf'(x) \qquad \text{for any } f \in C^2_{\text{pol}}.$$

c) Show that p_t preserves polynomials. Hence conclude that C^2_{pol} is a core for the generator.

Remark. *C^2_{pol} denotes the space of all twice continuously differentiable functions on $\mathbb{R}$ such that f, f' and f'' are growing at most polynomially at infinity.*

5. (Martingale problem for Feller processes). Let (p_t) be the transition function of a right-continuous time-homogeneous Markov process $((X_t)_{t\geq 0}, (P_x)_{x\in S})$ on a separable locally compact state space S such that

$$p_t\left(\hat{C}(S)\right) \subseteq \hat{C}(S) \qquad \forall\, t \geq 0.$$

a) Show that $(p_t)_{t\geq 0}$ induces a Feller semigroup, and (X_t, P_x) solves the martingale problem for the generator $(L, \mathrm{Dom}(L))$ for any $x \in S$.

b) Prove that for any $\alpha \geq 0$ and any $f \in \mathrm{Dom}(L)$,

$$M_t^{f,\alpha} \;=\; e^{-\alpha t} f(X_t) + \int_0^t e^{-\alpha s}\left(\alpha f - Lf\right)(X_s) ds$$

is a martingale.

6. (Differential operators as generators). Suppose that the generator of a Feller semigroup on $\mathbb{R}$ satisfies

$$(Lf)(x) \;=\; \sum_{n=0}^{m} a_n(x) \frac{d^n f}{dx^n}(x) \qquad \forall\, f \in C_0^\infty(\mathbb{R})$$

for some $m \in \mathbb{N}$ and coefficients $a_i \in C(\mathbb{R})$. Show that for any $x \in \mathbb{R}$,

$$a_0(x) \leq 0, \qquad a_2(x) \geq 0 \qquad \text{and} \qquad a_n(x) = 0 \;\; \forall\, n > 2.$$