Institute for Applied Mathematics
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"Markov Processes", Problem Sheet 13.

Exercises to be discussed in tutorials on Wednesday 1.2.

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1. (Stochastic dominance I). Let μ and ν be probability measures on $\mathbb{R}$. Prove that the following statements are equivalent.

(i) $\mu \preceq \nu$.

(ii) $F_\mu(c) = \mu((-\infty, c]) \geq F_\nu(c) \quad \forall\, c \in \mathbb{R}$.

(iii) There exists a coupling realized by random variables X and Y with distributions μ and ν, respectively, such that $X \leq Y$ almost surely.

2. (Stochastic dominance II). Let μ and ν be probability measures on the configuration space $S = \{0,1\}^{\mathbb{Z}^d}$, endowed with the product topology.

a) Prove that $\mu = \nu$ if and only if

$$\int f\, d\mu = \int f d\nu \quad \text{for any increasing, bounded and continuous function } f : S \to \mathbb{R}.$$

b) Conclude that if $\mu \preceq \nu$ and $\nu \preceq \mu$ then $\mu = \nu$.

3. (Ising model on $\mathbb{Z}^d$). Give a detailed proof of Theorem 4.15 in the lecture notes, i.e., show that for the Ising model, the maximal invariant probability measure $\bar{\mu}_\beta$ for the heat bath dynamics on $\mathbb{Z}^d$ coincides with the Gibbs measure μ_β^+ with "+ boundary conditions at infinity".

4. (Adjoint processes). Let p_t, $t \geq 0$, be the transition semigroup of a time-homogeneous Markov process with generator $\mathcal{L}$ on a *finite* state space S. Let μ be a probability measure with full support on S.

a) Write down explicitly the adjoint $\mathcal{L}^*$ of $\mathcal{L}$ as an operator in $L^2(S, \mu)$. Prove that $\mathcal{L}^*$ is the generator of a Markov process if and only if μ is invariant w.r.t. $(p_t)_{t\geq 0}$.

b) Show that in this case, the Markov process generated by $\mathcal{L}^*$ has the transition semigroup (p_t^*).

c) Give a probabilistic interpretation of this process when μ is the initial distribution.