Institute for Applied Mathematics
Winter term 2016/17
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"Markov Processes", Problem Sheet 12.

Hand in solutions before Monday 23.1., 2 pm

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1. (Infinitesimal characterization of invariant measures - A counterexample). Consider the minimal time-homogeneous Markov jump process (X_t, P_x) with state space $\mathbb{Z}$ and generator $\mathcal{L} = \lambda\,(\pi - I)$, where

$$\lambda(x) \;=\; 1 + x^2 \qquad \text{and} \qquad \pi(x,\cdot) \;=\; \delta_{x+1} \qquad \text{for any } x \in \mathbb{Z}.$$

a) Show that the probability measure μ with weights $\mu(x) \propto 1/(1+x^2)$ is infinitesimally invariant, i.e.,

$$(\mu\mathcal{L})(y) \;=\; 0 \qquad \text{for any } y \in \mathbb{Z}.$$

b) Show that nevertheless, μ is not an invariant measure for the transition semigroup (p_t) of the process.

2. (Decay of χ^2 divergence). Consider a time homogeneous Markov process on a finite state space S with generator $\mathcal{L}$ and invariant probability measure μ satisfying $\mu(x) > 0$ for any $x \in S$. Let ν_t denote the law at time t of the process started with an arbitrary initial law ν_0.

a) Compute the "carré du champ" operator

$$\Gamma(f,g) \;:=\; \mathcal{L}(f\cdot g) \,-\, f\,\mathcal{L}g \,-\, g\,\mathcal{L}f$$

for functions $f,g : S \to \mathbb{R}$, and verify that $\Gamma(f,f) \geq 0$.

b) Show that the χ^2 *divergence*

$$\chi^2(\nu_t|\mu) \;:=\; \sum_{x\in S} \left(\frac{\nu_t(x)}{\mu(x)} - 1\right)^2 \mu(x)$$

is a non-increasing function of t, and express the time-derivative in terms of the carré du champ.

c) Conclude that if there is a constant $\gamma > 0$ such that the *Poincaré inequality*

$$\mathrm{Var}_\mu(f) \;\leq\; \frac{1}{2\gamma}\int \Gamma(f,f)\,d\mu$$

holds for all functions $f : S \to \mathbb{R}$, then $\chi^2(\nu_t|\mu)$ is decaying exponentially with rate γ.