Institute for Applied Mathematics
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"Markov Processes", Problem Sheet 11.

Hand in solutions before Monday 16.1., 2 pm

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1. (Infinitesimal characterization of invariant measures). Consider a time-homogeneous continuous time Markov chain $X_t = Y_{N_t}$ where (N_t) is a Poisson process with constant intensity $\lambda > 0$, and (Y_n) is an independent Markov chain with transition matrix π on a finite state space S.

a) Show that the transition function is given by

$$p_t(x,y) \;=\; P_x[X_t = y] \;=\; \exp(t\mathcal{L})\,(x,y),$$

where $\mathcal{L} = \lambda(\pi - I)$ and $\exp(t\mathcal{L})$ is the matrix exponential. Hence conclude that $(p_t)_{t\geq 0}$ satisfies the forward and backward equation

$$\frac{d}{dt}p_t \;=\; p_t\mathcal{L} \;=\; \mathcal{L}p_t \qquad \text{for } t \geq 0.$$

b) Prove that a probability measure μ on S is invariant for (p_t) if and only if

$$\sum_{x\in S} \mu(x)\,\mathcal{L}(x,y) \;=\; 0 \qquad \text{for any } y \in S.$$

c) Show that the transition matrices are self-adjoint in $L^2(\mu)$, i.e.,

$$\sum_{x\in S} f(x)\,(p_t g)(x)\;\mu(x) \;=\; \sum_{x\in S} (p_t f)(x)\,g(x)\;\mu(x) \qquad \text{for any } t \geq 0,\;\; f,g : S \to \mathbb{R},$$

if and only if the generator $\mathcal{L}$ satisfies the detailed balance condition w.r.t. μ. What does this mean for the process ?

2. (Simple exclusion process). Let $\mathbb{Z}_n^d = \mathbb{Z}^d/(n\mathbb{Z})^d$ denote a discrete d-dimensional torus. The simple exclusion process on $S = \{0,1\}^{\mathbb{Z}_n^d}$ is the Markov process with generator

$$(\mathcal{L}f)\,(\eta) \;=\; \frac{1}{2d}\sum_{x\in\mathbb{Z}_n^d}\;\sum_{y:|y-x|=1} 1_{\{\eta(x)=1,\eta(y)=0\}}\cdot\left(f(\eta^{x,y}) - f(\eta)\right),$$

where $\eta^{x,y}$ is the configuration obtained from η by exchanging the values at x and y. Show that any Bernoulli measure of type

$$\mu_p \;=\; \bigotimes_{x\in\mathbb{Z}_n^d} \nu_p\,, \qquad \nu_p(1) = p,\;\; \nu_p(0) = 1-p,$$

$p \in [0,1]$, is invariant. Why does this not contradict the fact that any irreducible Markov process on a finite state space has a unique stationary distribution ?
(You may assume the statements of Exercise 1).

3. (Immigration-death process). Particles in a population die independently with rate $\mu > 0$. In addition, immigrants arrive with rate $\lambda > 0$. Assume that the population consists initially of one particle.

a) Explain why the population size X_t can be modeled by a birth-death process with rates $b(n) = \lambda$ and $d(n) = n\mu$.

b) Show that the generating function $G(s,t) = \mathbb{E}(s^{X_t})$ is given by

$$G(s,t) \;=\; \{1 + (s-1)e^{-\mu t}\} \exp\left\{\frac{\lambda}{\mu}(s-1)(1-e^{-\mu t})\right\}$$

c) Deduce the limiting distribution of X_t as $t \to \infty$.

4. (*A non-explosion criterion for jump processes). Suppose that $q_t(x,B) = \lambda_t(x)\pi_t(x,B)$ where π_t is a probability kernel on $(S,\mathcal{B})$ and $\lambda_t : S \to [0,\infty)$ is a measurable function. We consider the minimal jump process $((X_t), P_{t_0,x_0})$ with jump times J_n and positions Y_n defined by the following algorithm:

1) Set $J_0 := t_0$ and $Y_0 := x_0$.

2) For $n := 1, 2, \ldots$ do

   (i) Sample $E_n \sim \mathrm{Exp}(1)$ independently of $Y_0, \ldots, Y_{n-1}, E_0, \ldots, E_{n-1}$.

   (ii) Set $J_n := \inf\left\{t \geq 0 : \int_{J_{n-1}}^t \lambda_s(Y_{n-1})ds \geq E_n\right\}$.

   (iii) Sample $Y_n | (Y_0, \ldots, Y_{n-1}, E_0, \ldots, E_n) \sim \pi_{J_n}(Y_{n-1}, \cdot)$.

a) Explain why the construction coincides with the one in the lecture.

b) Show that if $\bar{\lambda} := \sup\limits_{t \geq 0} \sup\limits_{x \in S} \lambda_t(x) < \infty$, then the explosion time $\zeta = \sup J_n$ is almost surely infinite.

c) In the time-homogeneous case, given $\sigma(Y_k : k \in \mathbb{Z}_+)$,

$$J_n = \sum_{k=1}^{n} \frac{E_k}{\lambda(Y_{n-1})}$$

is a sum of conditionally independent exponentially distributed random variables. Conclude that the events

$$\{\zeta < \infty\} \qquad \text{and} \qquad \left\{\sum_{k=0}^{\infty} \frac{1}{\lambda(Y_k)} < \infty\right\}$$

coincide almost surely (apply Kolmogorov's 3-series Theorem).