Institute for Applied Mathematics
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"Markov Processes", Problem Sheet 9.

Hand in solutions before Monday 19.12., 2 pm
(post-box opposite to maths library)

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1. (Total variation distances).

a) Let $\nu = \bigotimes_{i=1}^d \nu_i$ and $\mu = \bigotimes_{i=1}^d \mu_i$ be two product probability measures on S^d. Show in at least two different ways that $\|\nu - \mu\|_{TV} \leq \sum_{i=1}^d \|\nu_i - \mu_i\|_{TV}$.

b) Show that the total variation distance of the law of a Markov chain to its stationary distribution is a non-increasing function of time.

c) Do similar statements as in a) and b) hold when the total variation distance is replaced by a general transportation metric $\mathcal{W}^1$?

Definition 1 *Let $\epsilon \in (0,1)$. The* **ϵ-mixing time** *of a Markov semigroup (p_t) with invariant probability measure μ is defined as* $t_{mix}(\epsilon) := \inf\{t \geq 0 : \|p_t(x,\cdot) - \mu\|_{TV} \leq \epsilon \ \forall x\}$.

2. (Hard core model). Consider a finite graph (V,E) with n vertices of maximal degree Δ. The corresponding hard core model with fugacity $\lambda > 0$ is the probability measure μ_λ on $\{0,1\}^V$ with mass function

$$\mu_\lambda(\eta) = \frac{1}{Z(\lambda)} \lambda^{\sum_{x\in V}\eta(x)} \quad \text{if } \eta(x)\cdot\eta(y) = 0 \text{ for any } (x,y) \in E, \quad \mu_\lambda(\eta) = 0 \quad \text{otherwise,}$$

where $Z(\lambda)$ is a normalization constant.

a) Describe the transition rule for the Glauber dynamics with equilibrium μ_λ, and determine the transition kernel π.

b) Prove that for $\lambda < (\Delta - 1)^{-1}$ and $t \in \mathbb{N}$,

$$\mathcal{W}^1(\nu\pi^t, \mu) \leq \alpha(n,\Delta)^t\, \mathcal{W}^1(\nu,\mu) \leq \exp\left(-\frac{t}{n}\left(\frac{1-\lambda(\Delta-1)}{1+\lambda}\right)\right) \mathcal{W}^1(\nu,\mu),$$

where $\alpha(n,\Delta) = 1 - \frac{1}{n}\left(\frac{1-\lambda(\Delta-1)}{1+\lambda}\right)$, and $\mathcal{W}^1$ is the transportation metric based on the Hamming distance on $\{0,1\}^V$.

c) Show that in this case, the ϵ-mixing time is of order $O(n \log n)$ for any $\epsilon \in (0,1)$.

3. (Conductance and lower bounds for mixing times). Let π be a transition kernel on $(S, \mathcal{B})$ with stationary distribution μ. For sets $A, B \in \mathcal{B}$ with $\mu(A) > 0$, the *equilibrium flow* $Q(A, B)$ from A to B is defined by

$$Q(A,B) \;=\; (\mu \otimes \pi)(A \times B) \;=\; \int_A \mu(dx)\, \pi(x, B),$$

and the *conductance* of A is given by

$$\Phi(A) \;=\; \frac{Q(A, A^C)}{\mu(A)}.$$

The *bottleneck ratio (isoperimetric constant)* Φ_* is defined as

$$\Phi_* = \min_{A:\mu(A)\leq 1/2} \Phi(A).$$

Let $\mu_A(B) = \mu(B|A)$ denote the conditioned measure on A.

a) Show that for any $A \in \mathcal{B}$ with $\mu(A) > 0$,

$$\|\mu_A \pi - \mu_A\|_{TV} \;=\; (\mu_A \pi)(A^C) \;=\; \Phi(A).$$

Hint: Prove first that

(i) $(\mu_A\pi)(B) - \mu_A(B) \leq 0$ *for any measurable* $B \subseteq A$, *and*

(ii) $(\mu_A\pi)(B) - \mu_A(B) = (\mu_A\pi)(B) \geq 0$ *for any measurable* $B \subseteq A^C$.

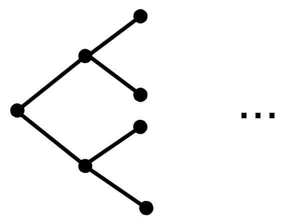
b) Conclude that

$$\|\mu_A - \mu\|_{TV} \leq t\Phi(A) + \|\mu_A \pi^t - \mu\|_{TV} \qquad \text{for any } t \in \mathbb{Z}_+.$$

c) Hence prove the lower bound

$$t_{mix}\left(\frac{1}{4}\right) \;\geq\; \frac{1}{4\Phi_*}.$$

4. (Lazy random walk on a binary tree).



Consider the lazy random walk with resting probability $\pi(x, x) = 1/2$ on a binary tree of depth k. Let $m = 2^{k+1} - 1$ denote the number of vertices. Prove that:

a) $t_{mix}(1/4) \;=\; O(m)$.

b) $t_{mix}(1/4) \;=\; \Omega(m)$.

Hint: You may assume the conductance bound from the previous exercise!