Institute for Applied Mathematics
Winter term 2016/17
Andreas Eberle, Raphael Zimmer

"Markov Processes", Problem Sheet 8.

Hand in solutions before Monday 12.12., 2 pm

---

1. (Asymptotic variances of ergodic averages). We consider a stationary Markov chain (X_n, P_μ) with state space $(S, \mathcal{B})$, transition kernel π, and initial distribution μ.

a) For $f \in \mathcal{L}^2(\mu)$ let $f_0 = f - \int f\,d\mu$, and let $A_t f = \frac{1}{t}\sum_{i=0}^{t-1} f(X_i)$. Prove (without assuming the CLT) that if $Gf_0 = \sum_{k=0}^{\infty} p^k f_0$ converges in $\mathcal{L}^2(\mu)$, then

$$\lim_{t\to\infty} t\,\mathrm{Var}\,[A_t f] \;=\; 2(f_0, Gf_0)_{L^2(\mu)} - (f_0, f_0)_{L^2(\mu)} \;=\; \mathrm{Var}_\mu(f) + \sum_{k=1}^{\infty} \mathrm{Cov}_\mu(f, \pi^k f).$$

b) Let $S = \{1, 2\}$, and suppose that the transition rates are given by $\pi(1,1) = \pi(2,2) = p$ and $\pi(2,1) = \pi(1,2) = 1-p$ with $p \in (0,1)$. Show that the unique stationary distribution μ is given by $\mu(1) = \mu(2) = 1/2$ for all values of p. Now consider

$$S_n \;=\; A_n - B_n,$$

where A_n and B_n are, respectively, the number of visits to the states 1 and 2 during the first n steps. Show that $S_n/\sqrt{n}$ satisfies a central limit theorem, and calculate the limiting variance as a function $\sigma^2(p)$ of p. How does $\sigma^2(p)$ behave as p tends to 0 or 1? Can you explain it? What is the value of $\sigma^2(1/2)$? Could you have guessed it?

2. (Random Walks on $\mathbb{Z}_+$). Let $\delta \in (0,1)$. We consider a random walk on the nonnegative integers with transition probabilities

$$\pi(x,y) = \begin{cases} 1/2 & \text{for } x = y, \\ (1-\delta)/4 & \text{for } y = x+1,\ x \ge 1, \\ (1+\delta)/4 & \text{for } y = x-1,\ x \ge 1 \\ 1/2 & \text{for } x = 0,\ y = 1. \end{cases}$$

a) Find the invariant probability measure $\mu(x)$ explicitly.

b) Let $f : \mathbb{Z}_+ \to \mathbb{R}$ be a function with compact support. Solve the Poisson equation $-\mathcal{L}g = f$ explicitly (e.g., by the ansatz $g = uh$ where h is a solution of $\mathcal{L}h = 0$). Show that for large x, a solution g either grows exponentially, or it is a constant.

c) Show that there is a solution g that is a constant for large x if and only if $\int f\,d\mu = 0$. What can you say about the asymptotic variance and the central limit theorem for $\sum_{j=0}^{n-1} f(X_j)$ for such functions f ?

3. (Couplings on $\mathbb{R}^d$). Let $W : \Omega \to \mathbb{R}^d$ be a random variable on $(\Omega, \mathcal{A}, P)$, and let μ_a denote the law of $a + W$.

a) (Synchronous coupling) Let $X = a + W$ and $Y = b + W$ for $a, b \in \mathbb{R}^d$. Show that

$$\mathcal{W}^2(\mu_a, \mu_b) \;=\; |a - b| \;=\; E(|X - Y|^2)^{1/2},$$

i.e., (X, Y) is an optimal coupling w.r.t. $\mathcal{W}^2$.

b) (Reflection coupling) Assume that $W \sim -W$. Let $\widetilde{Y} = \widetilde{W} + b$ where $\widetilde{W} = W - 2e \cdot W\, e$ with $e = \frac{a-b}{|a-b|}$. Prove that $(X, \tilde{Y})$ is a coupling of μ_a and μ_b, and if $|W| \le \frac{|a-b|}{2}$ a.s., then

$$E\left(f(|X - \widetilde{Y}|\right) \;\le\; f(|a-b|) \;=\; E\left(f(|X - Y|)\right)$$

for any concave, increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = 0$.

4. (Structure of invariant measures). Let π be a probability kernel on $(S, \mathcal{B})$, and let

$$\mathcal{S}(\pi) \;=\; \{\mu \in \mathcal{P}(S) : \mu = \mu\pi\}.$$

a) Show that $\mathcal{S}(\pi)$ is convex.

b) Prove that $\mu \in \mathcal{S}(\pi)$ is extremal if and only if every set $B \in \mathcal{B}$ such that $\pi 1_B = 1_B$ μ-a.e. satisfies $\mu(B) \in \{0, 1\}$.

c*) Show that every $\mu \in \mathcal{S}(\pi)$ is a convex combination of extremals.
(Hint: You may use part c) of Exercise 4 on Sheet 6.)