Institute for Applied Mathematics
Winter term 2016/17
Andreas Eberle, Raphael Zimmer

“Markov Processes”, Problem Sheet 7.

Asymptotic stationarity for Markov processes

Hand in solutions before Monday 5.12., 2 pm

---

1. (Equivalent characterizations of ergodicity for Markov processes). We consider a canonical right-continuous Markov process $((X_t)_{t\geq 0}, (P_x)_{x\in S})$ with Polish state space $(S, \mathcal{B})$ and transition semigroup $(p_t)_{t\geq 0}$. Let μ be a probability measure on $(S, \mathcal{B})$.

a) Show that P_μ is stationary if and only if μ is an invariant measure for (p_t).

b) From now on we assume that μ is an invariant probability measure for (p_t). Prove that the following nine conditions are all equivalent:

(i) P_μ is ergodic.

(ii) $\frac{1}{t}\int_0^t f(X_s)ds \to \int f d\mu$ P_μ-a.s. and in $L^2(P_\mu)$, for any $f \in \mathcal{L}^2(\mu)$.

(iii) $\mathrm{Var}_{P_\mu}\left[\frac{1}{t}\int_0^t f(X_s)\,ds\right] \to 0$ as $t \uparrow \infty$ for any $f \in \mathcal{L}^2(\mu)$.

(iv) $\frac{1}{t}\int_0^t \mathrm{Cov}_{P_\mu}\left[f(X_0), g(X_s)\right]\,ds \to 0$ as $t \uparrow \infty$ for any $f \in \mathcal{L}^2(\mu)$.

(v) $\frac{1}{t}\int_0^t P_\mu[X_0 \in B, X_s \in C]ds \to \mu(B)\mu(C)$ for any $B, C \in \mathcal{B}$.

(vi) $\frac{1}{t}\int_0^t p_s(x, B)ds \to \mu(B)$ $\mu-$a.e. for any $B \in \mathcal{B}$.

(vii) $P_x[T_B < \infty] > 0$ μ-a.e. for any $B \in \mathcal{B}$ such that $\mu(B) > 0$.

(viii) Every set $B \in \mathcal{B}$ such that $p_t 1_B = 1_B$ μ-a.e. for any $t \geq 0$ satisfies $\mu(B) \in \{0, 1\}$.

(ix) Every function $h \in \mathcal{L}^2(\mu)$ such that $p_t h = h$ μ-a.e. $\forall t \geq 0$ is almost surely constant.

c) A function $h : S \to \mathbb{R}$ is called *harmonic* w.r.t. (p_t) iff $p_t h = h$ for any $t \geq 0$. Show that the following statements are all equivalent:

(i) For any $x \in S$ and $B \in \mathcal{B}$ with $\mu(B) > 0$,

$$P_x\left[\forall\, s > 0\; \exists\, t \geq s : \; X_t \in B\right] \;=\; 1.$$

(ii) The constants are the only bounded harmonic functions for (p_t).

(iii) For any bounded measurable function $F : \Omega \to \mathbb{R}$ and any $x \in S$,

$$\frac{1}{t}\int_0^t F \circ \Theta_s\, ds \;\to\; E_\mu[F] \quad P_x\text{-almost surely.}$$

2. (Brownian motion on $\mathbb{R}/\mathbb{Z}$). A Brownian motion (X_t) on the circle $\mathbb{R}/\mathbb{Z}$ can be obtained by considering a Brownian motion (B_t) on $\mathbb{R}$ modulo the integers, i.e.,

$$X_t \;=\; B_t - \lfloor B_t \rfloor \;\in\; [0,1) \;\cong\; \mathbb{R}/\mathbb{Z}.$$

Prove the following statements:

a) Brownian motion on $\mathbb{R}/\mathbb{Z}$ is a Markov process with transition density w.r.t. the uniform distribution given by

$$p_t(x,y) \;=\; \frac{1}{\sqrt{2\pi t}} \sum_{n\in\mathbb{Z}} e^{-\frac{|x-y-n|^2}{2t}} \qquad \text{for any } t>0 \text{ and } x,y \in [0,1).$$

b) For any initial condition, (X_t) solves the martingale problem for the operator $\mathcal{L}f = f''/2$ defined on $C^\infty(\mathbb{R}/\mathbb{Z})$. *(Note that there is a one-to-one correspondence of smooth functions on $\mathbb{R}/\mathbb{Z}$ and periodic smooth functions on $\mathbb{R}$ with period 1).*

c) The uniform distribution μ is an invariant probability measure for (p_t), and the process with initial distribution μ is stationary and ergodic.

d) The generator $\mathcal{L}$ has smooth real-valued eigenfunctions e_n, $n\in\mathbb{Z}$, with corresponding eigenvalues $\lambda_n = 2\pi^2 n^2$. Moreover, $p_t e_n = \exp(-\lambda_n t) e_n$ for any $t \geq 0$.

e) For any $f \in \mathcal{L}^2(\mu)$,

$$\left\| p_t f - \int f\, d\mu \right\|_{L^2(\mu)} \;\leq\; e^{-2\pi^2 t}\, \mathrm{Var}_\mu(f).$$

f) Conclude that for the process with initial distribution μ,

$$E\left[\left(\frac{1}{t}\int_0^t f(X_s)\, ds - \int f\, d\mu\right)^2\right] \;\leq\; \frac{1}{\pi^2 t}\, \mathrm{Var}_\mu(f) \quad \text{for any } t\geq 0 \text{ and } f \in \mathcal{L}^2(\mu).$$

3. (Metropolis-Hastings method). Let $\mu(dx) = \mu(x)\, dx$ be a probability measure on $\mathbb{R}^d$ with strictly positive density, and let $q(x,dy) = q(x,y)\, dy$ be a probability kernel on $\mathbb{R}^d$ with strictly positive density. The Metropolis-Hastings acceptance probability is given by

$$\alpha(x,y) \;=\; \min\left(1, \frac{\mu(y)q(y,x)}{\mu(x)q(x,y)}\right), \qquad x,y \in \mathbb{R}^d.$$

Metropolis-Hastings algorithm
1.) Set $n := 0$ and choose some arbitrary point $X_0 \in \mathbb{R}^d$.
2.) Sample $Y_{n+1} \sim q(X_n, \cdot)$ and $U_{n+1} \sim \mathrm{Unif}(0,1)$ independently.
3.) If $U_{n+1} < \alpha(X_n, Y_{n+1})$ then set $X_{n+1} := Y_{n+1}$, else set $X_{n+1} := X_n$.
4.) Set $n := n+1$ and go to Step 2.

Show that for any bounded measurable function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\frac{1}{n}\sum_{i=0}^{n-1} f(X_i) \;\to\; \int f\, d\mu \qquad \text{almost surely.}$$