

“Markov Processes”, Problem Sheet 6.

Hand in solutions before Monday 28.11, 2 pm

1. (Ergodicity for stationary processes and Markov chains).

a) Let $((X_n)_{n \in \mathbb{Z}_+}, P)$ be a canonical stationary process with state space S , and let $\mathcal{J} = \{A : A = \Theta^{-1}(A)\}$ be the σ -algebra of shift invariant events. Prove that for $F : \Omega \rightarrow \mathbb{R}$ the following two properties are equivalent:

- (i) F is \mathcal{J} -measurable.
- (ii) $F = F \circ \Theta$.

Conclude that P is ergodic if and only if any shift-invariant function $F : \Omega \rightarrow \mathbb{R}$ is P -almost surely constant.

b) Now suppose that $((X_n), P_x)$ is a canonical time-homogeneous Markov chain, and let μ be an invariant probability measure for the transition kernel π . Show that the following properties are all equivalent:

- (i) P_μ is ergodic.
- (ii) $\frac{1}{n} \sum_{i=0}^{n-1} F \circ \Theta^i \rightarrow E_\mu[F]$ P_μ -almost surely for any $F \in \mathcal{L}^1(P_\mu)$.
- (iii) $\frac{1}{n} \sum_{i=0}^{n-1} f(X_i) \rightarrow \int f d\mu$ P_μ -almost surely for any $f \in \mathcal{L}^1(\mu)$.
- (iv) For any $B \in \mathcal{B}$,

$$\frac{1}{n} \sum_{i=0}^{n-1} \pi^i(x, B) \rightarrow \mu(B) \quad \text{for } \mu\text{-almost every } x.$$

- (v) For any $B \in \mathcal{B}$ such that $\mu(B) > 0$,

$$P_x[T_B < \infty] > 0 \quad \text{for } \mu\text{-almost every } x.$$

- (vi) Any set $B \in \mathcal{B}$ satisfying $\pi 1_B = 1_B$ μ -almost surely has measure $\mu(B) \in \{0, 1\}$.

Conclude that (i)-(vi) are satisfied if π is μ -irreducible.

2. (Rotations of the circle). Let $\Omega = \mathbb{R}/\mathbb{Z} = [0, 1]/\sim$ where $0 \sim 1$. We consider the rotation $\theta(\omega) = \omega + a \pmod{1}$ with $a = p/q$, $p, q \in \mathbb{N}$ relatively prime.

- a) Show that for any $x \in \Omega$, the uniform distribution P_x on $\{x, x + a, x + 2a, \dots, x + (q - 1)a\}$ is θ -invariant and ergodic.
- b) Determine all θ -invariant probability measures on Ω , and represent them as a mixture of ergodic ones.

3. (Ergodicity and decay of correlations). We consider a stationary stochastic process $(X_t)_{t \in [0, \infty)}$ defined on the canonical probability space (Ω, \mathcal{A}, P) .

a) Prove that the following properties are equivalent:

- (i) P is ergodic.
- (ii) $\text{Var} \left[\frac{1}{t} \int_0^t F \circ \theta_s ds \right] \rightarrow 0$ as $t \uparrow \infty$ for any $F \in \mathcal{L}^2(\Omega, \mathcal{A}, P)$.
- (iii) $\frac{1}{t} \int_0^t \text{Cov} [F \circ \theta_s, G] ds \rightarrow 0$ as $t \uparrow \infty$ for any $F, G \in \mathcal{L}^2(\Omega, \mathcal{A}, P)$.
- (iv) $\frac{1}{t} \int_0^t \text{Cov} [F \circ \theta_s, F] ds \rightarrow 0$ as $t \uparrow \infty$ for any $F \in \mathcal{L}^2(\Omega, \mathcal{A}, P)$.

b) The process (X_t) is said to be **mixing** iff

$$\lim_{t \rightarrow \infty} \text{Cov} (F \circ \theta_t, G) = 0 \quad \text{for any } F, G \in \mathcal{L}^2(\Omega, \mathcal{A}, P).$$

Prove that:

- (i) If $(X_t)_{t \geq 0}$ is mixing then it is ergodic.
- (ii) If the tail field $\mathcal{F} = \bigcap_{t \geq 0} \sigma(X_s : s \geq t)$ is trivial then $(X_t)_{t \geq 0}$ is mixing (and hence ergodic).

4. (Ergodicity and irreducibility for Markov processes in continuous time). We consider a canonical Markov process $((X_t)_{t \geq 0}, P_x)$ with state space (S, \mathcal{B}) and transition semigroup $(p_t)_{t \geq 0}$.

a) Show that for $\mu \in \mathcal{P}(S)$, the following three conditions are equivalent:

- (i) $P_\mu \circ \theta_t^{-1} = P_\mu$ for any $t \geq 0$.
- (ii) $((X_t)_{t \geq 0}, P_\mu)$ is a stationary process.
- (iii) μ is invariant with respect to p_t for any $t \geq 0$.

b) Show that the following three conditions are equivalent:

- (i) P_μ is ergodic.
- (ii) Every function $h \in \mathcal{L}^2(\mu)$ such that $p_t h = h$ μ -a.s. $\forall t \geq 0$ is almost surely constant.
- (iii) Every set $B \in \mathcal{B}$ such that $p_t 1_B = 1_B$ μ -a.s. for any $t \geq 0$ satisfies $\mu(B) \in \{0, 1\}$.

c) Show that for any shift-invariant event A , there exists $B \in \mathcal{B}$ with $p_t 1_B = 1_B$ μ -a.s. for any $t \geq 0$ such that

$$1_A = 1_{\{X_0 \in B\}} \quad P_\mu\text{-a.s.}$$