Institute for Applied Mathematics
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"Markov Processes", Problem Sheet 5.

Diffusion processes and Euler approximations

Hand in solutions before Monday 21.11, 2 pm

The goal of this problem sheet is to apply Lyapunov functions to study diffusion processes on $\mathbb{R}^d$ and Markov chains corresponding to Euler approximations of the corresponding stochastic differential equations.

Let $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be locally Lipschitz continuous functions. We consider a diffusion process (X_t, P_x) with possibly finite life-time ζ solving a stochastic differential equation

$$dX_t \;=\; \sigma(X_t)\,dB_t \,+\, b(X_t)\,dt \qquad \text{for } t<\zeta, \qquad X_0 = x, \tag{1}$$

where (B_t) is a d-dimensional Brownian motion. Let $a(x) \;=\; \sigma(x)\sigma(x)^T$. For the exercises below, it will only be important to know that (X_t, P_x) solves the local martingale problem for the generator

$$\mathcal{L}f \;=\; \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(x)\frac{\partial^2 f}{\partial x_i\,\partial x_j} \,+\, \sum_{i=1}^{d} b_i(x)\frac{\partial f}{\partial x_i}, \tag{2}$$

in the sense that for any $x \in \mathbb{R}^2$ and $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$,

$$M_t^f \;=\; f(t,X_t) - \int_0^t \left(\frac{\partial f}{\partial s} + \mathcal{L}f\right)(s,X_s)\,ds \tag{3}$$

is a local martingale w.r.t. P_x. More precisely, let $T_k \;=\; \inf\{t \geq 0 : |X_t| \geq k\}$. Then $\zeta = \sup T_k$, and for any $k \in \mathbb{N}$, the stopped process $(M_{t\wedge T_k})_{t\geq 0}$ is a martingale under P_x.

For a fixed time step $h > 0$, the *Euler-Maruyama approximation* of the diffusion process above is the time-homogeneous Markov chain (X_n^h, P_x) with transition step

$$x \;\mapsto\; x \,+\, \sqrt{h}\,\sigma(x)\,Z \,+\, h\,b(x), \qquad Z \sim N(0, I_d).$$

We denote the corresponding transition kernel and generator by π_h and $\mathcal{L}_h$, respectively.

1. (Explosions).

a) Prove that the diffusion process is non-explosive if

$$\operatorname{tr} a(x)/2 + x \cdot b(x) \;=\; O(|x|^2) \qquad \text{as } |x| \to \infty.$$

Show that for $\epsilon > 0$, the condition $O(|x|^2)$ can not be replaced by $O(|x|^{2+\epsilon})$.

b) Implement the Euler scheme on a computer, e.g. for $d = 1$. Do some experiments. Can you observe a different behavior in cases where the condition above is satisfied or violated, respectively ?

c) As discrete time Markov chains, the Euler approximations always have infinite lifetime. Which properties can you prove for the Euler approximations under similar conditions as in a) ?

2. (Stationary distributions I). Suppose that $\zeta = \infty$ almost surely, and that there exist $V \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$, $\varepsilon, c \in \mathbb{R}_+$, and a ball $B \subset \mathbb{R}^d$ such that $V \geq 0$ and

$$\frac{\partial V}{\partial t} + \mathcal{L}V \;\leq\; -\varepsilon + c1_B \qquad \text{on } \mathbb{R}_+ \times \mathbb{R}^d. \tag{4}$$

a) Prove that

$$E\left[\frac{1}{t}\int_0^t 1_B(X_s)\,ds\right] \;\geq\; \frac{\varepsilon}{c} - \frac{V(0,x_0)}{ct}.$$

b) It can be shown that (4) implies that (X_t, P_x) is a time-homogeneous Markov process with Feller transition semigroup $(p_t)_{t\geq 0}$. Conclude that there exists an invariant probability measure μ.
Hint: Try to carry over the proof in discrete time to the continuous time case.

3. (Stationary distributions II). Suppose that the conditions in Exercise 2 hold, and let μ be an invariant probability measure.

a) Show that $\int \mathcal{L}f\,d\mu = 0$ for any $f \in C_0^\infty(\mathbb{R}^d)$.

b) Use this to compute μ explicitly in the case $d = 1$. Here, assume that b and σ are twice continuously differentiable, and $\sigma(x) > 0$ for all x. You may also assume without proof that μ has a twice continuously differentiable density $\rho(x)$ which is strictly positive.

4. (Stationary distributions III).

a) Show that an invariant probability measure for the diffusion process on $\mathbb{R}^d$ exists if

$$\limsup_{|x|\to\infty} \left(\operatorname{tr} a(x)/2 \,+\, x \cdot b(x)\right) \;<0. \tag{5}$$

b) Give conditions ensuring the existence of an invariant probability measure for the Euler approximations. Why is not enough to assume (5) ?

c) Study different cases experimentally using an implementation of the Euler scheme. Can you see the difference between cases where an invariant probability measure exists or does not exist ?