Institute for Applied Mathematics
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"Markov Processes", Problem Sheet 4.

Hand in solutions before Monday 14.11, 2 pm
(post-box opposite to maths library)

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1. (Brownian motion reflected at 0). Let $(B_t)_{t\geq 0}$ be a standard one-dimensional Brownian motion with transition density $p_t(x, y)$.

a) Show that $X_t = |B_t|$ is a Markov process with transition density

$$p_t^+(x, y) \;=\; p_t(x, y) + p_t(x, -y).$$

b) Prove that (X_t, P) solves the martingale problem for the operator $\mathcal{L}f = \frac{1}{2}f''$ with domain

$$\mathcal{A} \;=\; \{f \in C_b^2([0, \infty)) : f'(0) = 0\}.$$

Hint: Note that functions in $\mathcal{A}$ can be extended to symmetric functions in $C_b^2(\mathbb{R})$.

c) Construct another solution to the martingale problem for $\mathcal{L}$ with domain $C_0^\infty(0, \infty)$. Does it also solve the martingale problem in b) ?

2. (Lyapunov functions and stochastic stability).

a) Consider a state space model on $\mathbb{R}^d$ with one-step transition $x \to x + b(x) + \sigma(x)W$, where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are measurable functions, and $W : \Omega \to \mathbb{R}^d$ is a square integrable random variable such that $E(W) = 0$ and $\mathrm{Cov}(W_i, W_j) = \delta_{ij}$. Show by considering a Lyapunov function $V(x) = |x|^2/\epsilon$ with $\epsilon > 0$, that sufficiently large balls are positive recurrent provided

$$\limsup_{|x|\to\infty} \left(2x \cdot b(x) + |b(x)|^2 + \mathrm{tr}(\sigma^T(x)\sigma(x))\right) \;<\; 0.$$

b) Consider a perturbed random walk on $\mathbb{Z}^d$ with transition rates

$$\pi(x, y) = \begin{cases} \frac{1}{2d} + \delta(x, y) & \text{for } |x - y| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find an estimate on the exit time from a ball of radius R. To this end consider for $\sigma > 0$ the function $F(x) = \exp\left(\sigma \sum_{i=1}^d |x_i|\right)$ on $\mathbb{Z}^d$ and show that

$$(\pi F)(x_1, \dots, x_d) \geq \theta F(x_1, \dots, x_d)$$

for some choices of $\sigma > 0$ and $\theta > 1$ that may depend on R.

3. (Recurrence on discrete state spaces). Let (X_n, P_x) be an irreducible homogeneous Markov chain with countable state space S.

a) Prove that the following three conditions are equivalent:

  i) There exists a finite set $A \subset S$ such that A is recurrent.

  ii) $\{x\}$ is recurrent for any $x \in S$.

  iii) $P_x(X_n = y \text{ infinitely often}) = 1 \; \forall \; x, y \in S$.

b) Show using Lyapunov functions that the simple random walk on $\mathbb{Z}^2$ is recurrent. (Hint: Consider for example the functions $V(x) = (\log(x))^\alpha$.)

4. (Tightness). Prove the following three statements.

a) A sequence of probability measures on the line is tight if and only if, for the corresponding distribution functions, we have $\lim_{x\to\infty} F_n(x) = 1$ and $\lim_{x\to-\infty} F_n(x) = 0$ uniformly in n.

b) A sequence of normal distributions on the line is tight if and only if the means and the variances are bounded (a normal distribution with variance 0 being a point mass).

c) A sequence of distributions of random variables X_n is tight if (X_n) is uniformly integrable.
Reminder: A sequence of random variables X_n is uniformly integrable if

$$\sup_{n\in\mathbb{N}} E[|X_n|; |X_n| \geq c] \to 0 \text{ as } c \to \infty.$$