Institute for Applied Mathematics
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"Markov Processes", Problem Sheet 3.

Hand in solutions before Monday 7.11, 2 pm
(post-box opposite to maths library)

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1. (Strong Markov property and Harris recurrence). Let (X_n, P_x) be a time homogeneous Markov chain on the state space $(S, \mathcal{B})$ with transition kernel $\pi(x, dy)$.

a) Show that if T is a finite $(\mathcal{F}_n^X)$ stopping time, then conditionally given $\mathcal{F}_T^X$, the process $\hat{X}_n := X_{T+n}$ is a Markov chain with transition kernel π starting in X_T.

b) Conclude that a set $A \in \mathcal{B}$ is Harris recurrent, i.e.,

$$P_x(X_n \in A \text{ for some } n \geq 1) = 1 \quad \text{for any } x \in A,$$

if and only if

$$P_x(X_n \in A \text{ infinitely often}) = 1 \quad \text{for any } x \in A.$$

2. (Passage times of the simple random walk). Let $S_n = \sum_{i=1}^n Z_i$ where $(Z_n)_{n\geq 1}$ are independent random variables with $P(Z_n = 1) = P(Z_n = -1) = 1/2$. Let a be a strictly positive integer, and let $T_a = \inf\{n \geq 0 : S_n = a\}$ be the first passage time to a.

a) Show that S_n and $S_n^2 - n$ are martingales. For $b < 0 < a$ compute $P[T_a < T_b]$ and $E[T_{\mathbb{Z}\setminus(a,b)}]$. Conclude that $E[T_a] = \infty$.

b) Show that for any $\theta \in \mathbb{R}$,

$$X_n^\theta \;=\; e^{\theta S_n}/(\cosh\theta)^n$$

is a martingale, and that for $\theta \geq 0$, $\left(X_{n\wedge T_a}^\theta\right)_{n\geq 0}$ is a bounded martingale that converges almost surely and in L^2 to the random variable

$$W^\theta \;=\; (\cosh\theta)^{-T_a} e^{\theta a}\, 1_{\{T_a<\infty\}}\,.$$

Conclude that $P(T_a < \infty) = 1$ and $E((\cosh\theta)^{-T_a}) = e^{-\theta a}$.

c) Explain how the results derived above can also be deduced from Theorem 1.2.

3. (Random walks on $\mathbb{Z}$). Let $((X_n)_{n\geq 0}, (P_x)_{x\in\mathbb{Z}})$ be the canonical Markov chain on $\mathbb{Z}$ with transition matrix Q given by

$$Q(x, x+1) = p,\ Q(x, x) = r,\ Q(x, x-1) = q$$

where $p + q + r = 1, p > 0, q > 0$ and $r \geq 0$. Fix $a, b \in \mathbb{Z}$ with $a < b - 1$ and let $T = \inf\{n \geq 0 : X_n \notin (a, b)\}$.

a) Prove that for any function $g : \{a+1, a+2, \ldots, b-1\} \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$, the system

$$\begin{aligned} &(Q - I)u(x) = -g(x), \quad a < x < b, \\ &u(a) = \alpha,\ u(b) = \beta, \end{aligned} \tag{1}$$

has a unique solution.

b) Conclude that $E_x(T) < \infty$ for any x. How can the mean exit time be computed explicitly ?

c) Assume for the moment that for every $s > 0$ and $x \in (a, b)$, $u_s(x) := E_x(T^s) < \infty$. Prove that u_2 is a solution of (1) for some α, β, g to be determined as functions of u_1.

d) Prove that there exists $\epsilon > 0$ such that $E_x[\exp(\lambda T)] < \infty$ for any $\lambda < \epsilon$. Hence conclude that $E_x(T^s) < \infty$ for every $s > 0$.

4. (Feynman-Kac formula). This exercise gives a direct proof of the uniqueness part in the Feynman-Kac formula. Let $((X_n)_{n\geq 0}, P_x)$ be a canonical time-homogeneous Markov chain with generator $\mathcal{L}$ on the state space S. Let $w : S \to \mathbb{R}_+$ be a nonnegative function.

a) For which functions v is

$$M_n = e^{-\sum_{k=0}^{n-1} w(X_i)} v(X_n)$$

a martingale?

b) Let $D \subset S$ be a measurable subset such that $T = \inf\{n > 0 : X_n \in D^c\} < \infty$ P_x-a.s. for any x, and let v be a bounded solution to the boundary value problem

$$\begin{aligned} &(\mathcal{L}v)(x) = (e^{w(x)} - 1)u(x) \quad \forall\, x \in D, \\ &v(x) = f(x) \quad \forall x \in D^c. \end{aligned} \tag{2}$$

Show by using a) that

$$v(x) \;=\; E_x\left(e^{-\sum_{k=0}^{T-1} w(X_k)} f(X_T)\right).$$