Institute for Applied Mathematics
Winter term 2016/17
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"Markov Processes", Problem Sheet 2.

Hand in solutions before Monday 31.10., 2 pm.
(post-box opposite to maths library)

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1. (Reduction to the time-homogeneous case). Let $I = \mathbb{R}_+$ or $I = \mathbb{Z}_+$.

a) Let $(X_t)_{t\in I}$ be a Markov process with transition function $(p_{s,t})$ and state space S. Show that for any $t_0 \in I$ the time-space process $\hat{X}_t = (t_0 + t, X_{t_0+t})$ is a time-homogeneous Markov process with state space $\mathbb{R}_+ \times S$ and transition function

$$\hat{p}_t\left((s,x),\cdot\right) = \delta_{s+t} \otimes p_{s,s+t}(x,\cdot).$$

b) Now suppose that $(X_t)_{t\in\mathbb{Z}_+}$ is a Markov chain with one step transition kernels π_t, $t \in \mathbb{N}$. Determine the one step transition kernel and the generator of the time-space process (t, X_t). Conclude that for any function $f \in \mathcal{F}_b(\mathbb{Z}_+ \times S)$, the process

$$M_t^{[f]} \; = \; f(t,X_t) - \sum_{k=0}^{t-1}(\mathcal{L}_k f(k+1,\cdot))(X_k) - \sum_{k=0}^{t-1}(f(k+1,X_k) - f(k,X_k))$$

is a martingale, where $(\mathcal{L}_t)$ are the generators of (X_t). What would be a corresponding statement in continuous time (without proof) ?

2. (Reflected Random Walks and Metropolis algorithm). Let π be a probability kernel on a measurable space $(S,\mathcal{B})$. A probability measure μ satisfies the *detailed balance condition* w.r.t. π iff for any $A, B \in \mathcal{B}$,

$$(\mu \otimes \pi)(A \times B) \;=\; (\mu \otimes \pi)(B \times A).$$

a) Prove that a probability measure μ which satisfies detailed balance w.r.t. π is also invariant w.r.t. π, i.e., $(\mu\pi)(B) = \int \mu(dx)\,\pi(x,B) = \mu(B)$ for any $B \in \mathcal{B}$.

b) Let $S \subset \mathbb{R}^d$ be a bounded measurable set. We define a Markov chain $(X_n)_{n\in\mathbb{Z}_+}$ by

$$X_0 \;=\; x_0 \in S, \qquad X_{n+1} \;=\; X_n + W_{n+1} \cdot 1_S(X_n + W_{n+1}),$$

where $W_i : \Omega \to \mathbb{R}^d$ are i.i.d. random variables. Suppose that the law of W_1 is absolutely continuous with a strictly positive density satisfying $f(x) = f(-x)$. Prove that the uniform distribution on S is invariant w.r.t. the transition kernel of the Markov chain (X_n).

c) Let $\mu(dx) = \mu(x)\,dx$ be a probability measure with strictly positive density on $\mathbb{R}^d$. Show that the process (X_n) defined by the following algorithm is a time-homogeneous Markov chain, identify its transition kernel π, and show that μ is invariant for π:

Random Walk Metropolis algorithm
1.) Set $n := 0$ and choose some arbitrary point $X_0 \in \mathbb{R}^d$.
2.) Set $Y_{n+1} := X_n + W_{n+1}$, and draw independently $U_{n+1} \sim \mathrm{Unif}(0,1)$.
3.) If $\mu(Y_{n+1})/\mu(X_n) > U_{n+1}$ then set $X_{n+1} := Y_{n+1}$, else set $X_{n+1} := X_n$.
4.) Set $n := n+1$ and go to Step 2.

3. (Recurrence of Brownian motion). A continuous-time stochastic process $((B_t)_{t\in[0,\infty)}, P_x)$ taking values in $\mathbb{R}^d$ is called a *Brownian motion starting at* x if the sample paths $t \mapsto B_t(\omega)$ are continuous, $B_0 = x$ P_x-a.s., and for every $f \in C_b^2(\mathbb{R}^d)$, the process

$$M_t^{[f]} \;:=\; f(B_t) - \frac{1}{2}\int_0^t \Delta f(B_s)\, ds$$

is a martingale w.r.t. the filtration $\mathcal{F}_t^B = \sigma(B_s : s \in [0,t])$. Let $T_a = \inf\{t \geq 0 : |B_t| = a\}$.

a) Compute $P_x[T_a < T_b]$ for $a < |x| < b$.

b) Show that for $d \leq 2$ a Brownian motion is recurrent in the sense that $P_x[T_a < \infty] = 1$ for any $a < |x|$.

c) Show that for $d \geq 3$ a Brownian motion is transient in the sense that $P_x[T_a < \infty] \to 0$ as $|x| \to \infty$.

You may assume the optional stopping theorem and the martingale convergence theorem in continuous time without proof. You may also assume that the Laplacian applied to a rotationally symmetric function $g(x) = \gamma(|x|)$ is given by

$$\Delta g(x) \;=\; r^{1-d}\frac{d}{dr}\left(r^{d-1}\frac{d}{dr}\gamma\right)(r) \;=\; \frac{d^2}{dr^2}\gamma(r) + \frac{d-1}{r}\frac{d}{dr}\gamma(r) \qquad \textit{where } r = |x|.$$

(How can you derive this expression rapidly if you do not remember it ?)

4. (Markov properties). Let $(X_t)_{t\in I}$ be a stochastic process with state space $(S_\Delta, \mathcal{B}_\Delta)$ defined on a probability space $(\Omega, \mathfrak{A}, P)$. Show that the following statements are equivalent:

(i) (X_t, P) is a Markov process with initial distribution ν and transition function $(p_{s,t})$.

(ii) For any $n \in \mathbb{Z}_+$ and $0 = t_0 \leq t_1 \leq \ldots \leq t_n$,

$$(X_{t_0}, X_{t_1}, \ldots, X_{t_n}) \sim \nu \otimes p_{t_0,t_1} \otimes p_{t_1,t_2} \otimes \cdots \otimes p_{t_{n-1},t_n} \qquad \text{w.r.t. } P.$$

(iii) $(X_t)_{t\in I} \sim P_\nu$.

(iv) For any $s \in I$, $P_{X_s}^{(s)}$ is a version of the conditional distribution of $(X_t)_{t\geq s}$ given $\mathcal{F}_s^X$, i.e.,

$$E[F((X_t)_{t\geq s})|\mathcal{F}_s^X] \;=\; E_{X_s}^{(s)}[F] \quad P\text{-a.s.}$$

for any product measurable function $F : S_\Delta^I \to \mathbb{R}_+$.

Here P_ν and $P_x^{(s)}$ denote the laws on S_Δ^I of the Markov processes with initial distributions ν, δ_x and transition functions $(p_{r,t})_{0\leq r\leq t}$, $(p_{s+r,s+t})_{0\leq r\leq t}$, respectively.