Institute for Applied Mathematics
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"Markov Processes", Problem Sheet 1.

Hand in solutions before Monday 24.10., 2 pm.
(post-box opposite to maths library)

Tutorial classes on wednesdays, 8-10 and 12-14, in 1.008.
First class: 19.10.

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1. (Conditional Expectations 1)
Let X, Y on $(\Omega, \mathcal{A}, P)$ be independent random variables that are Bernoulli distributed with parameter p. We set $Z = 1_{\{X+Y=0\}}$. Compute $E(X|Z)$ and $E(Y|Z)$. Are these random variables still independent?

2. (Conditional Expectations 2)

a) Let X, Y, Z be random variables with values in a measurable space $(S, \mathcal{B})$ such that the couples (X, Z) and (Y, Z) have the same law. Show that, for any $f \in \mathcal{F}_+(S)$,

$$E(f(X)|Z) = E(f(Y)|Z) \quad \text{a.s.}$$

b) Let $T_1, \ldots, T_n$ be i.i.d. real integrable random variables. Set $T = T_1 + \cdots + T_n$.

i) Show that $E(T_1|T) = T/n$.

ii) Compute $E(T|T_1)$.

3. (Conditional laws of Gaussian vectors)

a) Let X be a r.v. with values in $\mathbb{R}^m$ of the form $X = \phi(Y) + Z$, where Y and Z are independent. Show that the conditional law of X given $Y = y$ coincides with the law of $\phi(y) + Z$.

b) Let X, Y be random variables taking values in $\mathbb{R}^k$ and $\mathbb{R}^p$, respectively. We assume that their joint law in $(\mathbb{R}^{k+p}, \mathcal{B}(\mathbb{R}^{k+p}))$ is Gaussian with mean and covariance matrix given, respectively, by

$$\begin{pmatrix} m_X \\ m_Y \end{pmatrix}, \quad \begin{pmatrix} R_X & R_{XY} \\ R_{YX} & R_Y \end{pmatrix}.$$

Here m_X, m_Y, and R_X, R_Y are the means and covariance matrices of X and Y, respectively, and $R_{X,Y} = E((X - E(X))(Y - E(Y))^T) = R_{YX}^T$ is the $k \times p$ matrix of the covariances between components of X and Y. We assume that R_Y is strictly positive definite.

i) Find a $k \times p$ matrix A such that the random variables $X - AY$ and Y are independent.

(ii) Show that the conditional law of X given Y is Gaussian with mean $E(X|Y) = m_X + R_{XY}R_Y^{-1}(Y - m_Y)$ and covariance matrix $R_X - R_{XY}R_Y^{-1}R_{YX}$.

c) Let X be a signal with normal law $\mathcal{N}(0,1)$. We assume that we cannot observe the value of X; instead we only observe the value of $Y = X + W$, where W is independent of X with law $\mathcal{N}(0, \sigma^2)$.

i) Given an observation $Y = y$, give an estimate of the value X of the signal.

ii) Let us assume that $\sigma^2 = 0.1$ and that the value of the observation is $Y = 0.55$. What is the probability for the signal X to be in the interval $[\frac{1}{4}, \frac{3}{4}]$?

4. (Martingales)

a) Let $X = (X_n)_{n \in \mathbb{Z}_+}$ be a supermartingale such that $E(X_n)$ is constant. Show that X is a martingale.

b) Let $X = (X_n)_{n \in \mathbb{Z}_+}$ be an integrable process adapted to the filtration $(\mathcal{F}_n)$. Show that X is an $(\mathcal{F}_n)$-martingale if and only if $E(X_T) = E(X_0)$ for any bounded $(\mathcal{F}_n)$ stopping time T.

5. (Martingales with independent increments)

A process $(M_n)_{n \in \mathbb{Z}_+}$ is said to be with independent increments if, for every n, the r.v. $M_{n+1} - M_n$ is independent of the σ-algebra $\mathcal{F}_n = \sigma(M_0, \ldots, M_n)$.

a) Let $(M_n)_{n \in \mathbb{Z}_+}$ be a square integrable martingale with independent increments. We set $\sigma_0^2 = \mathrm{Var}(M_0)$ and, for $k \geq 1$, $\sigma_k^2 = \mathrm{Var}(M_k - M_{k-1})$.

i) Show that $\mathrm{Var}(M_n) = \sum_{k=0}^{n} \sigma_k^2$.

ii) Determine the conditional variance process $\langle M \rangle_n$ of (M_n).

b) Let (M_n) be a Gaussian martingale (we recall that a process $(M_n)_{n \geq 0}$ is Gaussian if, for every n, the vector $(M_0, \ldots, M_n)$ is Gaussian).

i) Show that $(M_n)_{n \geq 0}$ has independent increments.

ii) Show that, for every fixed $\theta \in \mathbb{R}$, the process $Z_n^\theta = \exp\left(\theta M_n - \frac{1}{2}\theta^2 \langle M \rangle_n\right)$ is a martingale. Does it converge almost surely?