Institut für angewandte Mathematik
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„Markov Processes", Problem Sheet 9

Please hand in your solutions before 12 noon on Monday, December 9.

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1. (Total variation distances).

a) Let $\nu = \bigotimes_{i=1}^{d} \nu_i$ and $\mu = \bigotimes_{i=1}^{d} \mu_i$ be two product probability measures on S^d. Show in at least two different ways that $\|\nu - \mu\|_{TV} \leq \sum_{i=1}^{d} \|\nu_i - \mu_i\|_{TV}$.

b) Show that the total variation distance of the law of a Markov chain to an invariant probability measure is a non-increasing function of time.

c) Do similar statements as in a) and b) hold when the total variation distance is replaced by a general transportation metric $\mathcal{W}^1$?

Definition 1 *Let $\epsilon \in (0,1)$. The* **ϵ-mixing time** *of a Markov semigroup (p_t) with invariant probability measure μ is defined as* $t_{mix}(\epsilon) := \inf\{t \geq 0 : \|p_t(x,\cdot) - \mu\|_{TV} \leq \epsilon \ \forall x\}$.

2. (Hard core model). Consider a finite graph (V, E) with n vertices of maximal degree Δ. The corresponding hard core model with fugacity $\lambda > 0$ is the probability measure μ_λ on $\{0,1\}^V$ with mass function

$$\mu_\lambda(\eta) \;=\; \frac{1}{Z(\lambda)}\,\lambda^{\sum_{x\in V}\eta(x)} \quad \text{if } \eta(x)\cdot\eta(y) = 0 \text{ for any } (x,y) \in E, \qquad \mu_\lambda(\eta) \;=\; 0 \quad \text{otherwise,}$$

where $Z(\lambda)$ is a normalization constant.

a) Describe the transition rule for the Gibbs sampler with equilibrium μ_λ, and determine the transition kernel π.

b) Prove that for $\lambda < (\Delta - 1)^{-1}$ and $t \in \mathbb{N}$,

$$\mathcal{W}^1(\nu\pi^t, \mu) \;\leq\; \alpha(n,\Delta)^t\, \mathcal{W}^1(\nu,\mu) \;\leq\; \exp\left(-\frac{t}{n}\left(\frac{1-\lambda(\Delta-1)}{1+\lambda}\right)\right) \mathcal{W}^1(\nu,\mu),$$

where $\alpha(n,\Delta) = 1 - \frac{1}{n}\left(\frac{1-\lambda(\Delta-1)}{1+\lambda}\right)$, and $\mathcal{W}^1$ is the transportation metric based on the Hamming distance on $\{0,1\}^V$.

c) Show that in this case, the ϵ-mixing time is of order $O(n \log n)$ for any $\epsilon \in (0,1)$.

3. (Bounds for ergodic averages in the non-stationary case). Let (X_n, P_x) be a Markov chain with transition kernel π and invariant probability measure μ, and let

$$A_{b,n}f \;=\; \frac{1}{n}\sum_{i=b}^{b+n-1} f(X_i).$$

Assume that there are a distance d on the state space S, and constants $\alpha \in (0,1)$ and $\bar{\sigma} \in \mathbb{R}_+$ such that

(A1) $\mathcal{W}_d^1(\nu\pi, \widetilde{\nu}\pi) \leq \alpha\, \mathcal{W}_d^1(\nu, \widetilde{\nu}) \;\; \forall\, \nu, \widetilde{\nu} \in \mathcal{P}(S)$, and

(A2) $\mathrm{Var}_{\pi(x,\cdot)}(f) \leq \bar{\sigma}^2\, \|f\|_{Lip(d)}^2 \;\; \forall\, x \in S,\; f : S \to \mathbb{R}$ Lipschitz.

Prove that under these assumptions the following bounds hold for any $b, n, k \geq 0$, $x \in S$, and for any Lipschitz continuous function $f : S \to \mathbb{R}$:

a) $\mathrm{Var}_x\left[f(X_n)\right] \;\leq\; \sum_{k=0}^{n-1} \alpha^{2k}\, \bar{\sigma}^2 \|f\|_{Lip(d)}^2$.

b) $\left|\mathrm{Cov}_x\left[f(X_n), f(X_{n+k})\right]\right| \;=\; \left|\mathrm{Cov}_x\left[f(X_n), (\pi^k f)(X_n)\right]\right| \;\leq\; \frac{\alpha^k}{1-\alpha^2}\bar{\sigma}^2 \|f\|_{Lip(d)}^2$.

c) $\mathrm{Var}_x\left[A_{b,n}f\right] \;\leq\; \frac{1}{n}\frac{\bar{\sigma}^2}{(1-\alpha)^2}\|f\|_{Lip(d)}^2$.

d) $\left|E_x\left[A_{b,n}f\right] - \mu(f)\right| \;\leq\; \frac{1}{n}\frac{\alpha^b}{1-\alpha}\int d(x,y)\,\mu(dy)\, \|f\|_{Lip(d)}$.

e) $E_x\left[\left|A_{b,n}f - \mu(f)\right|^2\right] \;\leq\; \frac{1}{n}\frac{1}{(1-\alpha)^2}\left(\bar{\sigma}^2 + \frac{1}{n}\alpha^{2b}\left(\int d(x,y)\,\mu(dy)\right)^2\right)\, \|f\|_{Lip(d)}^2$.