Institut für angewandte Mathematik
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„Markov Processes", Problem Sheet 8

Please hand in your solutions before 12 noon on Monday, December 2.

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1. (Couplings on $\mathbb{R}^d$). Let $W : \Omega \to \mathbb{R}^d$ be a random variable on $(\Omega, \mathcal{A}, P)$, and let μ_a denote the law of $a + W$.

a) Show that for $a, b \in \mathbb{R}^d$, every coupling of μ_a and μ_b can be realized by random variables $X = a + W_1$ and $Y = b + W_2$ with $W_1, W_2 \sim W$.

b) (Synchronous coupling) Let $X = a + W$ and $Y = b + W$. Show that (X, Y) is an optimal coupling of μ_a and μ_b w.r.t. $\mathcal{W}^2$.

c) (Reflection coupling) Now assume that the law of W is invariant under orthogonal transformations of $\mathbb{R}^d$, and let $\widetilde{Y} = b + \widetilde{W}$ where $\widetilde{W} = W - 2(e^T W)e$ with $e = \frac{a-b}{|a-b|}$. Prove that $(X, \tilde{Y})$ realizes a coupling of μ_a and μ_b, and if $|W| \leq \frac{|a-b|}{2}$ a.s., then

$$E\left[f(|X - \widetilde{Y}|)\right] \;\leq\; f(|a - b|) \;=\; E\left[f(|X - Y|)\right]$$

for any concave, increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f(0) = 0$.

2. (Wasserstein contractions). Let π be a probability kernel on a Polish space S. Suppose that $\overline{\pi}$ is a probability kernel on $S \times S$ such that $\overline{\pi}((x, y), dx'dy')$ is a coupling of $\pi(x, dx')$ and $\pi(y, dy')$ for any $x, y \in S$.

a) Prove that if there exists a distance function $d : S \times S \to [0, \infty)$ that is compatible with the topology on S and a constant $\alpha \in (0, 1)$ such that $\overline{\pi} d \leq \alpha d$, then there is a unique invariant probability measure μ of π, and

$$\mathcal{W}_d^1(\nu\pi^n, \mu) \;\leq\; \alpha^n \mathcal{W}_d^1(\nu, \mu) \qquad \text{for any } \nu \in \mathcal{P}^1(S).$$

Remark. If π is Feller, then the statement can be deduced from a result in the lecture. Try to give a direct proof without assuming the Feller property.

b) Apply the result to the autoregressive process on $\mathbb{R}^d$ with transition kernel

$$\pi(x, \bullet) \;=\; \mathcal{N}(\gamma x, I_d) \qquad \text{for some } \gamma \in (0, 1).$$

What is the invariant measure in this case ?

3. (Asymptotic variances of ergodic averages). We consider a stationary Markov chain (X_n, P_μ) with state space $(S, \mathcal{B})$, transition kernel π, and initial distribution μ.

a) For $f \in \mathcal{L}^2(\mu)$ let $f_0 = f - \int f \, d\mu$, and let $A_t f = \frac{1}{t}\sum_{i=0}^{t-1} f(X_i)$. Prove (without assuming the CLT) that if $Gf_0 = \sum_{k=0}^{\infty} \pi^k f_0$ converges in $\mathcal{L}^2(\mu)$, then

$$\lim_{t\to\infty} t \operatorname{Var}[A_t f] \;=\; 2(f_0, Gf_0)_{L^2(\mu)} - (f_0, f_0)_{L^2(\mu)} \;=\; \operatorname{Var}_\mu(f) + \sum_{k=1}^{\infty} \operatorname{Cov}_\mu(f, \pi^k f).$$

b) Let $S = \{1, 2\}$, and suppose that the transition rates are given by $\pi(1,1) = \pi(2,2) = p$ and $\pi(2,1) = \pi(1,2) = 1 - p$ with $p \in (0,1)$. Show that the unique stationary distribution μ is given by $\mu(1) = \mu(2) = 1/2$ for all values of p. Now consider

$$S_n \;=\; A_n - B_n,$$

where A_n and B_n are, respectively, the number of visits to the states 1 and 2 during the first n steps. Show that $S_n/\sqrt{n}$ satisfies a central limit theorem, and calculate the limiting variance as a function $\sigma^2(p)$ of p. How does $\sigma^2(p)$ behave as p tends to 0 or 1? Can you explain it? What is the value of $\sigma^2(1/2)$? Could you have guessed it?

4. (Structure of invariant measures). Let π be a probability kernel on $(S, \mathcal{B})$, and let

$$\mathcal{S}(\pi) \;=\; \{\mu \in \mathcal{P}(S) : \mu = \mu\pi\}.$$

a) Show that $\mathcal{S}(\pi)$ is convex.

b) Prove that $\mu \in \mathcal{S}(\pi)$ is extremal if and only if every set $B \in \mathcal{B}$ such that $\pi 1_B = 1_B$ μ-a.e. satisfies $\mu(B) \in \{0, 1\}$.

c) Show that every $\mu \in \mathcal{S}(\pi)$ is a convex combination of extremals.
(Hint: You may use Exercise 4 on Sheet 6.)