# „Markov Processes",  Problem Sheet 7

**On this sheet, Exercise 2 counts for two exercises.**
**Please hand in your solutions before 12 noon on Monday, November 25.**

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**1. (Equivalent characterizations of ergodicity for Markov processes).** We consider a canonical right-continuous Markov process $((X_t)_{t\geq 0}, (P_x)_{x\in S})$ with Polish state space $(S, \mathcal{B})$, transition semigroup $(p_t)_{t\geq 0}$, and invariant probability measure $\mu$.

a) Prove that the following nine conditions are all equivalent:

  (i) $P_\mu$ is ergodic.

  (ii) $\frac{1}{t}\int_0^t F \circ \Theta_s\, ds \to E_\mu[F]$ $P_\mu$-a.s. and in $L^2(P_\mu)$, for any $F \in \mathcal{L}^2(P_\mu)$.

  (iii) $\frac{1}{t}\int_0^t f(X_s) ds \to \int f d\mu$ $P_\mu$-a.s. and in $L^2(P_\mu)$, for any $f \in \mathcal{L}^2(\mu)$.

  (iv) $\frac{1}{t}\int_0^t \text{Cov}_{P_\mu}[f(X_0), g(X_s)]\, ds \to 0$  as $t \uparrow \infty$ for any $f, g \in \mathcal{L}^2(\mu)$.

  (v) $\frac{1}{t}\int_0^t P_\mu[X_0 \in B, X_s \in C] ds \to \mu(B)\mu(C)$ for any $B, C \in \mathcal{B}$.

  (vi) $\frac{1}{t}\int_0^t p_s(x, B) ds \to \mu(B)$ $\mu-$a.e. for any $B \in \mathcal{B}$.

  (vii) $P_x[\exists\, t \in \mathbb{Q}_+ : X_t \in B] > 0$ $\mu$-a.e. for any $B \in \mathcal{B}$ such that $\mu(B) > 0$.

  (viii) Every set $B \in \mathcal{B}$ such that $p_t 1_B = 1_B$ $\mu$-a.e. for any $t \geq 0$ satisfies $\mu(B) \in \{0, 1\}$.

  (ix) Every function $h \in \mathcal{L}^2(\mu)$ such that $p_t h = h$ $\mu$-a.e. $\forall t \geq 0$ is $\mu$-almost surely constant.

b) A function $h : S \to \mathbb{R}$ is called *harmonic* w.r.t. $(p_t)$ iff $p_t h = h$ for any $t \geq 0$. Show that the following statements are all equivalent:

  (i) For every $x \in S$ and $B \in \mathcal{B}$ with $\mu(B) > 0$,
$$P_x\left[\forall\, s > 0\, \exists\, t \geq s :\, X_t \in B\right]\; =\; 1.$$

  (ii) The constants are the only bounded harmonic functions for $(p_t)$.

  (iii) For every bounded measurable function $F : \Omega \to \mathbb{R}$ and every $x \in S$,
$$\frac{1}{t}\int_0^t F \circ \Theta_s\, ds\; \to\; E_\mu[F]\quad P_x\text{-almost surely.}$$

**2. (Brownian motion on $\mathbb{R}/\mathbb{Z}$).** A Brownian motion $(X_t)$ on the circle $\mathbb{R}/\mathbb{Z}$ can be obtained by considering a Brownian motion $(B_t)$ on $\mathbb{R}$ modulo the integers, i.e.,
$$X_t\; =\; B_t - \lfloor B_t \rfloor\; \in\; [0, 1)\; \cong\; \mathbb{R}/\mathbb{Z}.$$

Prove the following statements:

a) Brownian motion on $\mathbb{R}/\mathbb{Z}$ is a Markov process with transition density w.r.t. the uniform distribution given by

$$p_t(x, y) = \frac{1}{\sqrt{2\pi t}} \sum_{n \in \mathbb{Z}} e^{-\frac{|x-y-n|^2}{2t}} \qquad \text{for any } t > 0 \text{ and } x, y \in [0, 1).$$

b) For any initial condition, $(X_t)$ solves the martingale problem for the operator $\mathcal{L}f = f''/2$ defined on $C^\infty(\mathbb{R}/\mathbb{Z})$. *(Note that there is a one-to-one correspondence of smooth functions on $\mathbb{R}/\mathbb{Z}$ and periodic smooth functions on $\mathbb{R}$ with period 1).*

c) The uniform distribution $\mu$ is an invariant probability measure for $(p_t)$, and the process with initial distribution $\mu$ is stationary and ergodic.

d) The generator $-\mathcal{L}$ has smooth real-valued eigenfunctions $e_n$, $n \in \mathbb{Z}$, with corresponding eigenvalues $\lambda_n = 2\pi^2 n^2$. Moreover, $p_t e_n = \exp(-\lambda_n t) e_n$ for any $t \geq 0$.

e) For any $f \in \mathcal{L}^2(\mu)$,

$$\left\| p_t f - \int f \, d\mu \right\|_{L^2(\mu)} \leq e^{-2\pi^2 t} \sqrt{\mathrm{Var}_\mu(f)}.$$

f) Conclude that for the process with initial distribution $\mu$,

$$E\left[ \left( \frac{1}{t} \int_0^t f(X_s) \, ds - \int f \, d\mu \right)^2 \right] \leq \frac{1}{\pi^2 t} \mathrm{Var}_\mu(f) \quad \text{for any } t \geq 0 \text{ and } f \in \mathcal{L}^2(\mu).$$

**3. (Metropolis-Hastings method).** Let $\mu(dx) = \mu(x) \, dx$ be a probability measure on $\mathbb{R}^d$ with strictly positive density, and let $q(x, dy) = q(x, y) \, dy$ be a probability kernel on $\mathbb{R}^d$ with strictly positive density. The Metropolis-Hastings acceptance probability is given by

$$\alpha(x, y) = \min\left( 1, \frac{\mu(y)q(y, x)}{\mu(x)q(x, y)} \right), \qquad x, y \in \mathbb{R}^d.$$

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**Metropolis-Hastings algorithm**
1.) Set $n := 0$ and choose some arbitrary point $X_0 \in \mathbb{R}^d$.
2.) Sample $Y_{n+1} \sim q(X_n, \cdot)$ and $U_{n+1} \sim \mathrm{Unif}(0, 1)$ independently.
3.) If $U_{n+1} < \alpha(X_n, Y_{n+1})$ then set $X_{n+1} := Y_{n+1}$, else set $X_{n+1} := X_n$.
4.) Set $n := n + 1$ and go to Step 2.

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Show that for every bounded measurable function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\frac{1}{n} \sum_{i=0}^{n-1} f(X_i) \to \int f \, d\mu \qquad \text{almost surely.}$$

*The student council of mathematics will organize the Maths Party on 28/11 in N8schicht. The presale will be held on Tue 26/11, Wed 27/11 and Thu 28/11 in the mensa Poppelsdorf. Further information can be found on fsmath.uni-bonn.de.*