Institut für angewandte Mathematik
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Andreas Eberle

UNIVERSITÄT BONN

„Markov Processes", Problem Sheet 6

Please hand in your solutions before 12 noon on Monday, November 18.

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1. (Ergodicity for stationary processes and Markov chains).

a) Let $((X_n)_{n\in\mathbb{Z}_+}, P)$ be a canonical stationary process with state space S, and let $\mathcal{J} = \{A : A = \Theta^{-1}(A)\}$ be the σ-algebra of shift invariant events. Prove that for $F : \Omega \to \mathbb{R}$ the following two properties are equivalent:

(i) F is $\mathcal{J}$- measurable.

(ii) $F = F \circ \Theta$.

Conclude that P is ergodic if and only if any shift-invariant function $F : \Omega \to \mathbb{R}$ is P-almost surely constant.

b) Now suppose that $((X_n), P_x)$ is a canonical time-homogeneous Markov chain, and let μ be an invariant probability measure for the transition kernel π. Show that the following properties are all equivalent:

(i) P_μ is ergodic.

(ii) $\frac{1}{n}\sum_{i=0}^{n-1} F \circ \Theta^i \to E_\mu[F]$ $\quad P_\mu$-almost surely for any $F \in \mathcal{L}^1(P_\mu)$.

(iii) $\frac{1}{n}\sum_{i=0}^{n-1} f(X_i) \to \int f\, d\mu$ $\quad P_\mu$-almost surely for any $f \in \mathcal{L}^1(\mu)$.

(iv) For any $B \in \mathcal{B}$,

$$\frac{1}{n}\sum_{i=0}^{n-1} \pi^i(x,B) \to \mu(B) \qquad \text{for } \mu\text{-almost every } x.$$

(v) For any $B \in \mathcal{B}$ such that $\mu(B) > 0$,

$$P_x[T_B < \infty] > 0 \qquad \text{for } \mu\text{-almost every } x.$$

(vi) Any set $B \in \mathcal{B}$ satisfying $\pi 1_B = 1_B$ μ-almost surely has measure $\mu(B) \in \{0, 1\}$.

2. (Rotations of the circle). Let $\Omega = \mathbb{R}/\mathbb{Z} = [0,1]/\sim$ where $0 \sim 1$. We consider the rotation $\theta(\omega) = \omega + a \pmod 1$ with $a = p/q$, $p, q \in \mathbb{N}$ relatively prime.

a) Show that for every $x \in \Omega$, the uniform distribution P_x on $\{x, x+a, x+2a, \dots, x+(q-1)a\}$ is θ-invariant and ergodic.

b) Determine all θ-invariant probability measures on Ω, and represent them as a mixture of ergodic ones.

3. (Ergodicity and decay of correlations). We consider a stationary stochastic process $(X_t)_{t\in[0,\infty)}$ defined on the canonical probability space $(\Omega, \mathcal{A}, P)$.

a) Prove that the following properties are equivalent:

   (i) P is ergodic.

   (ii) $\mathrm{Var}\left(\frac{1}{t}\int_0^t F\circ\theta_s\, ds\right) \to 0$ as $t \uparrow \infty$ for any $F \in \mathcal{L}^2(\Omega, \mathcal{A}, P)$.

   (iii) $\frac{1}{t}\int_0^t \mathrm{Cov}\,(F\circ\theta_s, G)\, ds \to 0$ as $t\uparrow\infty$ for any $F, G \in \mathcal{L}^2(\Omega, \mathcal{A}, P)$.

   (iv) $\frac{1}{t}\int_0^t \mathrm{Cov}\,(F\circ\theta_s, F)\, ds \to 0$ as $t\uparrow\infty$ for any $F \in \mathcal{L}^2(\Omega, \mathcal{A}, P)$.

b) The process (X_t) is said to be **mixing** iff

$$\lim_{t\to\infty} \mathrm{Cov}\,(F\circ\theta_t\,,\, G) \;=\; 0 \qquad \text{for any } F, G \in \mathcal{L}^2(\Omega, \mathcal{A}, P).$$

   Prove that:

   (i) If $(X_t)_{t\geq 0}$ is mixing then it is ergodic.

   (ii) If the tail field $\mathcal{F} = \bigcap_{t\geq 0} \sigma(X_s : s \geq t)$ is trivial then $(X_t)_{t\geq 0}$ is mixing (and hence ergodic).

4. (Ergodicity and irreducibility for Markov processes in continuous time). We consider a canonical Markov process $((X_t)_{t\geq 0}, P_x)$ with state space $(S, \mathcal{B})$ and transition semigroup $(p_t)_{t\geq 0}$.

a) Show that for $\mu \in \mathcal{P}(S)$, the following three conditions are equivalent:

   (i) $P_\mu \circ \theta_t^{-1} = P_\mu$ for any $t \geq 0$.

   (ii) $((X_t)_{t\geq 0}, P_\mu)$ is a stationary process.

   (iii) μ is invariant with respect to p_t for all $t \geq 0$.

b) Show that the following three conditions are equivalent:

   (i) P_μ is ergodic.

   (ii) Every function $h \in \mathcal{L}^2(\mu)$ such that $p_t h = h$ μ-a.s. $\forall t \geq 0$ is almost surely constant.

   (iii) Every set $B \in \mathcal{B}$ such that $p_t 1_B = 1_B$ μ-a.s. for any $t \geq 0$ satisfies $\mu(B) \in \{0, 1\}$.

c) Show that for every shift-invariant event A, there exists $B \in \mathcal{B}$ with $p_t 1_B = 1_B$ μ-a.s. for all $t \geq 0$ such that

$$1_A \;=\; 1_{\{X_0 \in B\}} \qquad P_\mu\text{-a.s.}$$