Institut für angewandte Mathematik
Wintersemester 2019/20
Andreas Eberle

„Markov Processes", Problem Sheet 5

Please hand in your solutions before 12 noon on Monday, November 11, into the marked post boxes opposite to the maths library.

The goal of this problem sheet is to apply Lyapunov functions to study diffusion processes on $\mathbb{R}^d$ and Markov chains corresponding to Euler approximations of the corresponding stochastic differential equations.

Let $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be locally Lipschitz continuous functions. We consider a diffusion process (X_t, P_x) with possibly finite life-time ζ solving a stochastic differential equation

$$dX_t \; = \; \sigma(X_t)\, dB_t \, + \, b(X_t)\, dt \qquad \text{for } t < \zeta, \qquad X_0 = x, \tag{1}$$

where (B_t) is a d-dimensional Brownian motion. Let $a(x) \; = \; \sigma(x)\sigma(x)^T$. For the exercises below, it will only be important to know that (X_t, P_x) solves the local martingale problem for the generator

$$\mathcal{L}f \; = \; \frac{1}{2}\sum_{i,j=1}^d a_{i,j}(x)\frac{\partial^2 f}{\partial x_i\, \partial x_j} \, + \, \sum_{i=1}^d b_i(x)\frac{\partial f}{\partial x_i}, \tag{2}$$

in the sense that for any $x \in \mathbb{R}^2$ and $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$,

$$M_t^f \; = \; f(t, X_t) \, - \, \int_0^t \left(\frac{\partial f}{\partial s} + \mathcal{L}f\right)(s, X_s)\, ds \tag{3}$$

is a local martingale w.r.t. P_x. More precisely, let $T_k \; = \; \inf\{t \geq 0 : |X_t| \geq k\}$. Then $\zeta = \sup T_k$, and for any $k \in \mathbb{N}$, the stopped process $(M_{t\wedge T_k})_{t\geq 0}$ is a martingale under P_x.

For a fixed time step $h > 0$, the *Euler-Maruyama approximation* of the diffusion process above is the time-homogeneous Markov chain (X_n^h, P_x) with transition step

$$x \;\mapsto\; x \, + \, \sqrt{h}\, \sigma(x)\, Z \, + \, h\, b(x), \qquad Z \sim N(0, I_d).$$

We denote the corresponding transition kernel and generator by π_h and $\mathcal{L}_h$, respectively.

1. (Explosions).

a) Prove that the diffusion process is non-explosive if

$$\operatorname{tr} a(x)/2 + x \cdot b(x) \;=\; O(|x|^2) \qquad \text{as } |x| \to \infty.$$

Show that for $\epsilon > 0$, the condition $O(|x|^2)$ can not be replaced by $O(|x|^{2+\epsilon})$.

b) Implement the Euler scheme on a computer, e.g. for $d = 1$. Do some experiments. Can you observe a different behavior in cases where the condition above is satisfied or violated, respectively ?

c) As discrete time Markov chains, the Euler approximations always have infinite lifetime. Which properties can you prove for the Euler approximations under similar conditions as in a) ?

2. (Stationary distributions I). Suppose that $\zeta = \infty$ almost surely, and that there exist $V \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$, $\varepsilon, c \in \mathbb{R}_+$, and a ball $B \subset \mathbb{R}^d$ such that $V \geq 0$ and

$$\frac{\partial V}{\partial t} + \mathcal{L}V \;\leq\; -\varepsilon + c 1_B \qquad \text{on } \mathbb{R}_+ \times \mathbb{R}^d. \tag{4}$$

a) Prove that

$$E\left[\frac{1}{t}\int_0^t 1_B(X_s)\,ds\right] \;\geq\; \frac{\varepsilon}{c} - \frac{V(0,x_0)}{ct}.$$

b) It can be shown that (4) implies that (X_t, P_x) is a time-homogeneous Markov process with Feller transition semigroup $(p_t)_{t\geq 0}$. Conclude that there exists an invariant probability measure μ.
Hint: Try to carry over the proof in discrete time to the continuous time case.

3. (Stationary distributions II). Suppose that the conditions in Exercise 2 hold, and let μ be an invariant probability measure.

a) Show that $\int \mathcal{L}f\,d\mu = 0$ for any $f \in C_0^\infty(\mathbb{R}^d)$.

b) Use this to compute μ explicitly in the case $d = 1$. Here, assume that b and σ are twice continuously differentiable, and $\sigma(x) > 0$ for all x. You may also assume without proof that μ has a twice continuously differentiable density $\rho(x)$ which is strictly positive.

4. (Stationary distributions III).

a) Show that an invariant probability measure for the diffusion process on $\mathbb{R}^d$ exists if

$$\limsup_{|x|\to\infty} \;(\operatorname{tr} a(x)/2 \,+\, x \cdot b(x)) \;<0. \tag{5}$$

b) Give conditions ensuring the existence of an invariant probability measure for the Euler approximations. Why is not enough to assume (5) ?

c) Study different cases experimentally using an implementation of the Euler scheme. Can you see the difference between cases where an invariant probability measure exists or does not exist ?