Institut für angewandte Mathematik
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Andreas Eberle

UNIVERSITÄT BONN

„Markov Processes", Problem Sheet 4

Please hand in your solutions before 12 noon on Monday, November 4, into the marked post boxes opposite to the maths library.

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1. (Recurrence on discrete state spaces). Let (X_n, P_x) be an irreducible homogeneous Markov chain with countable state space S.

a) Prove that the following three conditions are equivalent:

(i) There exists a finite set $A \subset S$ such that A is recurrent.

(ii) $\{x\}$ is recurrent for any $x \in S$.

(iii) $P_x(X_n = y \text{ infinitely often}) = 1 \ \forall\, x, y \in S$.

b) Show using Lyapunov functions that the simple random walk on $\mathbb{Z}^2$ is recurrent. *Hint: Consider for example the functions* $V(x) = (\log |x|)^\alpha$.

2. (Lyapunov functions and stochastic stability).

a) Consider a state space model on $\mathbb{R}^d$ with one-step transition $x \to x + b(x) + \sigma(x)W$, where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are measurable functions, and $W : \Omega \to \mathbb{R}^d$ is a square integrable random variable such that $E(W) = 0$ and $\mathrm{Cov}(W_i, W_j) = \delta_{ij}$. Show by considering a Lyapunov function $V(x) = |x|^2/\epsilon$ with $\epsilon > 0$, that sufficiently large balls are positive recurrent provided

$$\limsup_{|x|\to\infty} \left(2x \cdot b(x) + |b(x)|^2 + \mathrm{tr}(\sigma^T(x)\sigma(x))\right) \ < \ 0.$$

b) Consider a perturbed random walk on $\mathbb{Z}^d$ with transition rates

$$\pi(x, y) = \begin{cases} \frac{1}{2d} + \delta(x, y) & \text{for } |x - y| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find an estimate on the exit time from a ball of radius R. To this end consider for $\sigma > 0$ the function $F(x) = \exp\left(\sigma \sum_{i=1}^d |x_i|\right)$ on $\mathbb{Z}^d$ and show that

$$(\pi F)(x_1, \dots, x_d) \geq \theta F(x_1, \dots, x_d)$$

for some choices of $\sigma > 0$ and $\theta > 1$ that may depend on R.

3. (Tightness). Prove the following three statements.

a) A sequence of probability measures on the line is tight if and only if, for the corresponding distribution functions, we have $\lim_{x\to\infty} F_n(x) = 1$ and $\lim_{x\to-\infty} F_n(x) = 0$ uniformly in n.

b) A sequence of normal distributions on the line is tight if and only if the means and the variances are bounded (a normal distribution with variance 0 being a point mass).

c) A sequence of distributions of random variables X_n is tight if (X_n) is uniformly integrable.
Reminder: A sequence of random variables X_n is uniformly integrable if

$$\sup_{n\in\mathbb{N}} E[|X_n|; |X_n| \geq c] \to 0 \text{ as } c \to \infty.$$