Institut für angewandte Mathematik
Wintersemester 2019/20
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„Markov Processes", Problem Sheet 3

Please hand in your solutions before 12 noon on Monday, October 28, into the marked post boxes opposite to the maths library.

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1. (Strong Markov property and Harris recurrence). Let (X_n, P_x) be a time homogeneous $(\mathcal{F}_n)$ Markov chain on the state space $(S, \mathcal{B})$ with transition kernel $\pi(x, dy)$.

a) Show that if T is a finite $(\mathcal{F}_n)$ stopping time, then conditionally given $\mathcal{F}_T$, the process $\hat{X}_n := X_{T+n}$ is a Markov chain with transition kernel π starting in X_T.

b) Conclude that a set $A \in \mathcal{B}$ is *Harris recurrent*, i.e.,

$$P_x[X_n \in A \text{ for some } n \geq 1] = 1 \quad \text{for any } x \in A,$$

if and only if

$$P_x[X_n \in A \text{ infinitely often}] = 1 \quad \text{for any } x \in A.$$

2. (Strong Markov property in continuous time). Suppose that (X_t, P_x) is a time homogeneous $(\mathcal{F}_t)$ Markov process in continuous time with state space $\mathbb{R}^d$ and transition semigroup (p_t).

a) Let T be an $(\mathcal{F}_t)$ stopping time taking only the discrete values $t_i = ih$, $i \in \mathbb{Z}_+$, for some fixed $h \in (0, \infty)$. Prove that for any initial value $x \in \mathbb{R}^d$ and any non-negative measurable function $F : (\mathbb{R}^d)^{[0,\infty)} \to \mathbb{R}$,

$$E_x\left[F(X_{T+\bullet})|\mathcal{F}_T\right] \;=\; E_{X_T}[F(X)] \qquad P_x\text{-almost surely.} \tag{1}$$

b) The transition semigroup (p_t) is called *Feller* iff for every $t \geq 0$ and every bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$, $x \mapsto (p_t f)(x)$ is continuous. Prove that if $t \mapsto X_t(\omega)$ is right continuous for all ω and (p_t) is a Feller semigroup, then the strong Markov property (1) holds for every $(\mathcal{F}_t)$ stopping time $T : \Omega \to [0, \infty)$.
Hint: Show first that for any $t \geq 0$ and $f \in C_b(\mathbb{R}^d)$,

$$E_x\left[f(X_{T+t})|\mathcal{F}_T\right] \;=\; E_{X_T}[f(X_t)] \qquad P_x\text{-almost surely.} \tag{2}$$

3. (Brownian motion killed at 0). Let $X_t = B_t$ for $t < T$ and $X_t = \Delta$ for $t \geq T$, where $(B_t)_{t\geq 0}$ is a one-dimensional Brownian motion, and $T = \inf\{t \geq 0 : B_t = 0\}$.

a) Show that $(X_t)_{t\geq 0}$ is a Markov process on the extended state space $(0,\infty)\dot{\cup}\{\Delta\}$ with transition kernel satisfying $p_t^{\mathrm{Dir}}(x,B) = \int_B p_t^{\mathrm{Dir}}(x,y)\,dy$ for any $x \in (0,\infty)$ and $B \in \mathcal{B}((0,\infty))$, where

$$p_t^{\mathrm{Dir}}(x,y) \;=\; \frac{1}{\sqrt{2\pi t}}\left(\exp\left(-\frac{(y-x)^2}{2t}\right) - \exp\left(-\frac{(y+x)^2}{2t}\right)\right) \quad \text{for } x,y \in (0,\infty).$$

b) We extend functions $f : (0,\infty) \to \mathbb{R}$ to the extended state space $(0,\infty)\dot{\cup}\{\Delta\}$ by setting $f(\Delta) := 0$, Prove that in this sense, (X_t, P) solves the martingale problem for the operator $\mathcal{L}f = \frac{1}{2}f''$ with domain

$$\mathcal{A} \;=\; \{f \in C_b^2([0,\infty)) : f(0) = 0\}.$$

4. (Boundary value problems for Markov chains). Let (X_n, P_x) be a canonical time-homogeneous Markov chain with measurable state space $(S,\mathcal{B})$ and generator $\mathcal{L}$.

a) The first part of this exercise gives a direct proof of the uniqueness part in the Feynman-Kac formula for Markov chains. Let $w : S \to \mathbb{R}$ be a non-negative measurable function. Determine for which functions v the process

$$M_n \;=\; e^{-\sum_{k=0}^{n-1} w(X_i)} v(X_n)$$

is a martingale. Now let $D \subset S$ be a measurable subset such that

$$T = \inf\{n \geq 0 : X_n \in D^c\} < \infty \quad P_x\text{-a.s. for any } x,$$

and let v be a bounded solution to the boundary value problem

$$\begin{aligned} (\mathcal{L}v)(x) &= (e^{w(x)} - 1)v(x) \quad \text{for all } x \in D, \\ v(x) &= f(x) \quad \text{for all } x \in D^c. \end{aligned} \tag{3}$$

Show that

$$v(x) \;=\; E_x\left[e^{-\sum_{k=0}^{T-1} w(X_k)} f(X_T)\right].$$

b) Suppose that $S = \mathbb{Z}$ and the transition matrix π of (X_n, P_x) is given by

$$\pi(x,x+1) = p,\; \pi(x,x) = r,\; \pi(x,x-1) = q$$

where $p+q+r = 1$, $p > 0$, $q > 0$ and $r \geq 0$. Fix $a,b \in \mathbb{Z}$ with $a < b-1$ and let

$$T = \inf\{n \geq 0 : X_n \notin (a,b)\}.$$

Prove that for any function $g : \{a+1, a+2, \ldots, b-1\} \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$, the system

$$\begin{aligned} (\mathcal{L}u)(x) &= -g(x) \quad \text{for } a < x < b, \\ u(a) &= \alpha, \quad u(b) \;=\; \beta, \end{aligned} \tag{4}$$

has a unique solution. Conclude that $E_x[T] < \infty$ for any x. How can the mean exit time be computed explicitly ?