Institut für angewandte Mathematik
Wintersemester 2019/20
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„Markov Processes", Problem Sheet 2

Please hand in your solutions before 12 am on Monday, October 21, into the marked post boxes opposite to the maths library.

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1. (Reduction to the time-homogeneous case). Let $I = \mathbb{R}_+$ or $I = \mathbb{Z}_+$.

a) Let $(X_t)_{t\in I}$ be a Markov process with transition function $(p_{s,t})$ and state space S. Show that for any $t_0 \in I$, the time-space process $\hat{X}_t = (t_0 + t, X_{t_0+t})$ is a time-homogeneous Markov process with state space $\mathbb{R}_+ \times S$ and transition function

$$\hat{p}_t\left((s,x),\cdot\right) = \delta_{s+t} \otimes p_{s,s+t}(x,\cdot).$$

b) Now suppose that $(X_t)_{t\in\mathbb{Z}_+}$ is a Markov chain with one step transition kernels π_t, $t \in \mathbb{N}$. Determine the one step transition kernel and the generator of the time-space process (t, X_t). Conclude that for any function $f \in \mathcal{F}_b(\mathbb{Z}_+ \times S)$, the process

$$M_t^{[f]} \;=\; f(t,X_t) \;-\; \sum_{k=0}^{t-1} \mathcal{L}_k(f(k+1,\cdot))(X_k) \;-\; \sum_{k=0}^{t-1} (f(k+1,X_k) - f(k,X_k))$$

is a martingale, where $(\mathcal{L}_t)$ are the generators of (X_t). What would be a corresponding statement in continuous time (without proof) ?

2. (Markov properties).

a) Let $(X_t)_{t\in\mathbb{R}_+}$ be an arbitrary stochastic process on a probability space $(\Omega, \mathcal{A}, P)$. Show that the *historical process* $\hat{X}_t = (X_s)_{s\in[0,t]}$ is an $(\mathcal{F}_t^X)$ Markov process.

b) Let (B_t) be a standard Brownian motion on a probability space $(\Omega, \mathcal{A}, P)$, and let

$$X_t \;=\; \int_0^t B_s\, ds \qquad \text{and} \qquad S_t \;=\; \sup_{s\le t} B_s.$$

Show that (X_t) and (S_t) are not Markov processes, but the two component processes (B_t, X_t) and (B_t, S_t) are Markov.

c) Determine the laws of these Markov processes at time t.

3. (Brownian motion reflected at 0). Let $(B_t)_{t\geq 0}$ be a standard one-dimensional Brownian motion with transition density $p_t(x,y)$.

a) Show that $X_t = |B_t|$ is a Markov process with transition density

$$p_t^{\text{refl}}(x,y) \;=\; \frac{1}{\sqrt{2\pi t}}\left(\exp\left(-\frac{(y-x)^2}{2t}\right)+\exp\left(-\frac{(y+x)^2}{2t}\right)\right).$$

b) Prove that (X_t, P) solves the martingale problem for the operator $\mathcal{L}f = \frac{1}{2}f''$ with domain

$$\mathcal{A} \;=\; \{f \in C_b^2([0,\infty)) : f'(0) = 0\}.$$

Hint: Note that functions in $\mathcal{A}$ can be extended to symmetric functions in $C_b^2(\mathbb{R})$.

c) Construct another solution to the martingale problem for $\mathcal{L}$ with domain $C_0^\infty(0,\infty)$. Does it also solve the martingale problem in b) ?

4. (Reflected Random Walks and Metropolis algorithm). Let π be a probability kernel on a measurable space $(S,\mathcal{B})$. A probability measure μ satisfies the *detailed balance condition* w.r.t. π iff for any $A, B \in \mathcal{B}$,

$$(\mu\otimes\pi)(A\times B) \;=\; (\mu\otimes\pi)(B\times A).$$

a) Prove that a probability measure μ which satisfies detailed balance w.r.t. π is also invariant w.r.t. π, i.e., $(\mu\pi)(B) = \int \mu(dx)\,\pi(x,B) = \mu(B)$ for any $B\in\mathcal{B}$.

b) Let $S\subset\mathbb{R}^d$ be a bounded measurable set. We define a Markov chain $(X_n)_{n\in\mathbb{Z}_+}$ by

$$X_0 \;=\; x_0\in S, \qquad X_{n+1} \;=\; X_n + W_{n+1}\cdot 1_S(X_n+W_{n+1}),$$

where $W_i : \Omega \to \mathbb{R}^d$ are i.i.d. random variables. Suppose that the law of W_1 is absolutely continuous with a strictly positive density satisfying $f(x) = f(-x)$. Prove that the uniform distribution on S is invariant w.r.t. the transition kernel of the Markov chain (X_n).

c) Let $\mu(dx) = \mu(x)\,dx$ be a probability measure with strictly positive density on $\mathbb{R}^d$. Show that the process (X_n) defined by the following algorithm is a time-homogeneous Markov chain, identify its transition kernel π, and show that μ is invariant for π:

Random Walk Metropolis algorithm
1.) Set $n := 0$ and choose some arbitrary point $X_0\in\mathbb{R}^d$.
2.) Set $Y_{n+1} := X_n + W_{n+1}$, and draw independently $U_{n+1}\sim \mathrm{Unif}(0,1)$.
3.) If $\mu(Y_{n+1})/\mu(X_n) > U_{n+1}$ then set $X_{n+1} := Y_{n+1}$, else set $X_{n+1} := X_n$.
4.) Set $n := n+1$ and go to Step 2.