Institut für angewandte Mathematik
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„Markov Processes", Problem Sheet 13

Please hand in your solutions before 12 noon on Monday, January 20.

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1. (Simple exclusion process). Let $\mathbb{Z}_n^d = \mathbb{Z}^d / (n\mathbb{Z})^d$ denote a discrete d-dimensional torus. The simple exclusion process on $S = \{0,1\}^{\mathbb{Z}_n^d}$ is the continuous time Markov process with generator

$$(\mathcal{L}f)(\eta) \;=\; \frac{1}{2d} \sum_{x \in \mathbb{Z}_n^d} \sum_{y : |y-x|=1} 1_{\{\eta(x)=1, \eta(y)=0\}} \cdot \left(f(\eta^{x,y}) - f(\eta) \right),$$

where $\eta^{x,y}$ is the configuration obtained from η by exchanging the values at x and y. Show that every Bernoulli measure of type

$$\mu_p \;=\; \bigotimes_{x \in \mathbb{Z}_n^d} \nu_p\,, \qquad \nu_p(1) = p,\; \nu_p(0) = 1-p,$$

$p \in [0,1]$, is invariant. Why does this not contradict the fact that an irreducible Markov process on a finite state space has a unique stationary distribution ?

2. (Infinitesimal characterization of invariant measures - A counterexample). Consider the minimal time-homogeneous Markov jump process (X_t, P_x) with state space $\mathbb{Z}$ and generator $\mathcal{L} = \lambda\,(\pi - I)$, where

$$\lambda(x) \;=\; 1 + x^2 \qquad \text{and} \qquad \pi(x,\cdot) \;=\; \delta_{x+1} \qquad \text{for all } x \in \mathbb{Z}.$$

a) Show that the probability measure μ with weights $\mu(x) \propto 1/(1+x^2)$ is infinitesimally invariant, i.e.,

$$(\mu\mathcal{L})(y) \;=\; 0 \qquad \text{for all } y \in \mathbb{Z}.$$

b) Show that nevertheless, μ is not an invariant measure for the transition semigroup (p_t) of the process.

3. (Stochastic dominance I). Let μ and ν be probability measures on $\mathbb{R}$. Prove that the following statements are equivalent.

(i) $\mu \preceq \nu$, i.e., $\int f \, d\mu \leq \int f \, d\nu$ for every increasing function $f \in \mathcal{F}_b(\mathbb{R})$.

(ii) $F_\mu(c) = \mu((-\infty, c]) \geq F_\nu(c) \quad \forall \, c \in \mathbb{R}$.

(iii) There exists a coupling realized by random variables X and Y with distributions μ and ν, respectively, such that $X \leq Y$ almost surely.

4. (Stochastic dominance II). Let μ and ν be probability measures on the configuration space $S = \{0,1\}^{\mathbb{Z}^d}$, endowed with the product topology.

a) Prove that $\mu = \nu$ if and only if

$$\int f \, d\mu = \int f d\nu \quad \text{for any increasing, bounded and continuous function } f : S \to \mathbb{R}.$$

b) Conclude that if $\mu \preceq \nu$ and $\nu \preceq \mu$ then $\mu = \nu$.