Institut für angewandte Mathematik
Wintersemester 2019/20
Andreas Eberle

„Markov Processes", Problem Sheet 12

Please hand in your solutions before 12 noon on Monday, January 13.

---

1. (Strong continuity of transition semigroups of Markov processes on L^p spaces). Suppose that $(p_t)_{t\geq 0}$ is the transition function of a *right-continuous*, time homogeneous Markov process $((X_t)_{t\geq 0}, (P_x)_{x\in S})$, and $\mu \in \mathcal{M}_+(S)$ is a sub-invariant measure.

a) Show that for every $f \in C_b(S)$ and $x \in S$,

$$(p_t f)(x) \to f(x) \qquad \text{as } t \downarrow 0.$$

b) Now let f be a non-negative function in $C_b(S) \cap \mathcal{L}^1(S,\mu)$ and $p \in [1,\infty)$. Show that as $t \downarrow 0$,

$$\int |p_t f - f| d\mu \ \to \ 0\,, \qquad \text{and hence} \qquad p_t f \to f \text{ in } L^p(S,\mu).$$

Hint: You may use that $|x| = x + 2x^-$.

c) Conclude that (p_t) induces a strongly continuous contraction semigroup of linear operators on $L^p(S,\mu)$ for every $p \in [1,\infty)$.

2. (Semigroups generated by self-adjoint operators on Hilbert spaces). Show that if E is a Hilbert space (for example an L^2 space) with norm $\|f\| = (f,f)^{1/2}$, and L is a *self-adjoint* linear operator, i.e.,

$$(L, \mathrm{Dom}(L)) = (L^*, \mathrm{Dom}(L^*)),$$

then L is the generator of a C^0 contraction semigroup on E if and only if L is *negative definite*, i.e.,

$$(f, Lf) \leq 0 \qquad \text{for all } f \in \mathrm{Dom}(L).$$

Remark. In this case, the C^0 semigroup generated by L is given by $P_t = e^{tL}$, where the exponential is defined by spectral theory, see e.g. Reed & Simon: Methods of modern mathematical physics, Vol. I and II.

3. (Brownian motion with absorption at 0). Brownian motion with absorption at 0 is the Markov process with state space $S = [0, \infty)$ defined by $X_t = B_{t \wedge T_0}$ where (B_t, P_x) is a Brownian motion on $\mathbb{R}$.

a) On which Banach spaces does this process induce C^0 contraction semigroups ?

b) Identify the corresponding generators.

4. (Approximation of semigroups by resolvents). Suppose that $(P_t)_{t \geq 0}$ is a Feller semigroup with resolvent $(G_\alpha)_{\alpha > 0}$.

a) Prove that for any $g \in \hat{C}(S)$, $t > 0$, $n \in \mathbb{N}$ and $x \in S$,

$$\left(\left(\frac{n}{t} G_{\frac{n}{t}} \right)^n g \right)(x) \;=\; E\left[\left(P_{\frac{E_1 + \cdots + E_n}{n} t}\, g \right)(x) \right]$$

where $(E_k)_{k \in \mathbb{N}}$ is a sequence of independent exponentially distributed random variables with parameter 1.

b) Hence conclude that

$$\left(\frac{n}{t} G_{\frac{n}{t}} \right)^n g \to P_t g \quad \text{uniformly as } n \to \infty. \tag{1}$$

c) How could you derive (1) more directly if the state space is finite?