Institut für angewandte Mathematik
Wintersemester 2019/20
Andreas Eberle

UNIVERSITÄT BONN

„Markov Processes", Problem Sheet 11

Please hand in your solutions before 12 noon on Tuesday, January 7.

We wish you a merry Christmas and a happy new year!

1. (Strongly continuous semigroups and resolvents).

a) State the defining properties of a strongly continuous contraction semigroup and a strongly continuous contraction resolvent on a Banach space E.

b) Prove that if (P_t) is a C_0 contraction semigroup then $G_\alpha f = \int_0^\infty e^{-\alpha t} P_t f \, dt$ defines a C_0 contraction resolvent.

2. (Uniform motion to the right). Consider the deterministic Markov process (X_t, P_x) on $\mathbb{R}$ given by $X_t = x + t$ P_x-almost surely.

a) Show that the transition semigroup $(P_t)_{t\geq 0}$ is strongly continuous both on $\hat{C}(\mathbb{R})$ and on $L^2(\mathbb{R}, dx)$.

b) Prove that the generator on $\hat{C}(\mathbb{R})$ is given by

$$Lf \;=\; f' , \qquad \mathrm{Dom}(L) \;=\; \{f \in C^1(\mathbb{R}) : f, f' \in \hat{C}(\mathbb{R})\} .$$

c) Show that the generator on $L^2(\mathbb{R}, dx)$ is given by

$$Lf \;=\; f' , \qquad \mathrm{Dom}(L) \;=\; H^{1,2}(\mathbb{R}, dx).$$

3. (Ornstein-Uhlenbeck process). The transition semigroup of the Ornstein-Uhlenbeck process on $\mathbb{R}$ is given by

$$(p_t f)(x) \;=\; (2\pi)^{-1/2} \int f\left(e^{-t}x + \sqrt{1-e^{-2t}}\, y\right)\, e^{-y^2/2}\, dy \qquad \text{for } f \in \mathcal{F}_b(\mathbb{R}).$$

a) Show that the standard normal distribution γ is invariant.

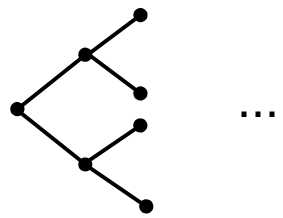
b) Let L denote the generator on $L^2(\mathbb{R},\gamma)$. Show that $C^2_{\text{pol}} \subset \text{Dom}(L)$ and

$$(Lf)(x) \;=\; f''(x) - xf'(x) \qquad \text{for any } f \in C^2_{\text{pol}}.$$

c) Show that p_t preserves polynomials. Hence conclude that C^2_{pol} is a core for the generator.

Remark. C^2_{pol} *denotes the space of all twice continuously differentiable functions on* $\mathbb{R}$ *such that* f, f' *and* f'' *are growing at most polynomially at infinity.*

4. (Lazy random walk on a binary tree). Consider the lazy random walk with resting probability $\pi(x,x) = 1/2$ on a binary tree of depth k.



Let $m = 2^{k+1} - 1$ denote the number of vertices, and let T be the first hitting time of the root. Prove that:

a) $t_{mix}(1/4) \;=\; \Omega(m)$.

b) $\sup_x E_x[T] \;=\; O(m)$.

c) $t_{mix}(1/4) \;=\; O(m)$.