Institut für angewandte Mathematik
Wintersemester 2019/20
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„Markov Processes", Problem Sheet 10

Please hand in your solutions before 12 noon on Monday, December 16.

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1. (Conductance and lower bounds for mixing times). Let π be a transition kernel on $(S, \mathcal{B})$ with invariant probability measure μ. For sets $A, B \in \mathcal{B}$ with $\mu(A) > 0$, the *equilibrium flow* $Q(A, B)$ from A to B is defined by

$$Q(A,B) \;=\; (\mu \otimes \pi)(A \times B) \;=\; \int_A \mu(dx)\,\pi(x,B),$$

and the *conductance* of A is given by

$$\Phi(A) \;=\; \frac{Q(A,A^C)}{\mu(A)}\,.$$

The *bottleneck ratio (isoperimetric constant)* Φ_* is defined as

$$\Phi_* = \min_{A:\mu(A)\leq 1/2} \Phi(A).$$

Let $\mu_A(B) = \mu(B|A)$ denote the conditioned measure given A.

a) Show that for any $A \in \mathcal{B}$ with $\mu(A) > 0$,

$$\|\mu_A\pi - \mu_A\|_{TV} \;=\; (\mu_A\pi)(A^C) \;=\; \Phi(A).$$

Hint: Prove first that

(i) $(\mu_A\pi)(B) - \mu_A(B) \leq 0$ *for any measurable* $B \subseteq A$, *and*

(ii) $(\mu_A\pi)(B) - \mu_A(B) = (\mu_A\pi)(B) \geq 0$ *for any measurable* $B \subseteq A^C$.

b) Conclude that

$$\|\mu_A - \mu\|_{TV} \leq t\Phi(A) + \|\mu_A\pi^t - \mu\|_{TV} \qquad \text{for any } t \in \mathbb{Z}_+.$$

c) Hence prove the lower bound

$$t_{mix}\left(\frac{1}{4}\right) \;\geq\; \frac{1}{4\Phi_*}\,.$$

2. (Gibbs sampler for the Ising model). Consider a finite graph (V, E) with n vertices of maximal degree Δ. The Ising model with inverse temperature $\beta \geq 0$ is the probability measure μ_β on $\{-1, 1\}^V$ with mass function

$$\mu_\beta(\eta) \;=\; \frac{1}{Z(\beta)} \exp\left(\beta \sum_{\{x,y\}\in E} \eta(x)\eta(y)\right),$$

where $Z(\beta)$ is a normalization constant.

a) Show that given $\eta(y)$ for $y \neq x$, $\eta(x) = \pm 1$ with probability $(1 \pm \tanh(\beta m(x, \eta))/2$, where $m(x, \eta) := \sum_{y\sim x} \eta(y)$ is the local magnetization in the neighbourhood of x. Hence determine the transition kernel π for the Gibbs sampler with equilibrium μ_β.

b) Prove that for any $t \in \mathbb{N}$,

$$\mathcal{W}^1(\nu\pi^t, \mu_\beta) \;\leq\; \alpha(n, \beta, \Delta)^t\, \mathcal{W}^1(\nu, \mu_\beta) \;\leq\; \exp\left(-\frac{t}{n}(1 - \Delta \tanh(\beta))\right) \mathcal{W}^1(\nu, \mu_\beta),$$

where $\alpha(n, \beta, \Delta) = 1 - (1 - \Delta \tanh(\beta))/n$, and $\mathcal{W}^1$ is the transportation metric based on the Hamming distance on $\{-1, 1\}^V$. Conclude that for $\Delta \tanh \beta < 1$, the Gibbs sampler is geometrically ergodic with a rate of order $\Omega(1/n)$.
Hint: You may use the inequality

$$|\tanh(y + \beta) - \tanh(y - \beta)| \;\leq\; 2 \tanh(\beta) \qquad \text{for any } \beta \geq 0 \text{ and } y \in \mathbb{R}.$$

c) The *mean-field Ising model* with parameter $\alpha \geq 0$ is the Ising model on the complete graph over $V = \{1, \ldots, n\}$ with inverse temperature $\beta = \alpha/n$. Show that for $\alpha < 1$, the ϵ-mixing time for the Gibbs sampler on the mean field Ising model is of order $O(n \log n)$ for any $\epsilon \in (0, 1)$.

3. (Successful couplings and TV-convergence to equilibrium). Consider a time-homogeneous Markov chain with transition kernel π and stationary distribution μ.

a) Show that for every initial distribution ν and every fixed integer $t \geq 0$, there exists a coupling (X, Y, P) of the Markov chains with transition kernel π and initial laws ν and μ such that

$$P\left[X_n = Y_n \text{ for all } n \geq t\right] \;=\; 1 - \left\|\nu\pi^t - \mu\right\|_{TV}.$$

b) Conclude that $\|\nu\pi^t - \mu\|_{TV} \to 0$ as $t \to \infty$ if and only if for every $\epsilon > 0$ there exists a coupling of the Markov chains with initial laws ν and μ such that the coupling time

$$T \;=\; \inf\{t \geq 0 \,:\, X_n = Y_n \text{ for all } n \geq t\}$$

is finite with probability at least $1 - \epsilon$.

***c) A coupling as above is called *successful* if the coupling time is almost surely finite. Show that a successful coupling exists if and only if $\|\nu\pi^t - \mu\|_{TV} \to 0$ as $t \to \infty$.
(This part is optional and very difficult.)