Institut für angewandte Mathematik
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„Markov Processes", Problem Sheet 0

The exercises on this sheet will be discussed in the tutorials during the first week. You are encouraged to study them in advance but you do not have to submit written solutions.

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1. (Revision of conditional expectations 1). Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, $\mathcal{F} \subset \mathcal{A}$ a σ-algebra, and $X : \Omega \to \mathbb{R}_+$ a non-negative random variable.

a) Define the conditional expectation $\mathbb{E}[X|\mathcal{F}]$.

b) Suppose that there exists a decomposition of Ω into disjoint sets $A_1, \dots A_n$ such that $\mathcal{F} = \sigma(\{A_1, \dots, A_n\})$. Show that

$$\mathbb{E}[X|\mathcal{F}] \;=\; \sum\nolimits_{i:\mathbb{P}[A_i]>0} \mathbb{E}[X|A_i]\, 1_{A_i}$$

is a version of the conditional expectation of X given $\mathcal{F}$.

2. (Revision of conditional expectations 2). Let $X, Y : \Omega \to \mathbb{R}_+$ be non-negative random variables. Show that $\mathbb{P}$-almost surely, the following identities hold:

a) For $\lambda \in \mathbb{R}$ we have $\mathbb{E}[\lambda X + Y|\mathcal{F}] = \lambda\mathbb{E}[X|\mathcal{F}] + \mathbb{E}[Y|\mathcal{F}]$.

b) $\mathbb{E}[\mathbb{E}[X|\mathcal{F}]] = \mathbb{E}[X]$ and $|\mathbb{E}[X|\mathcal{F}]| \leq \mathbb{E}[|X|\,|\mathcal{F}]$.

c) If $\sigma(X)$ is independent of $\mathcal{F}$, then $\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X]$.

d) Let $(S, \mathcal{S})$ and $(T, \mathcal{T})$ be measurable spaces. If $Y : \Omega \to S$ is $\mathcal{F}$-measurable, $X : \Omega \to T$ is independent of $\mathcal{F}$ and $f : S \times T \to [0, \infty)$ is product-measurable, then

$$\mathbb{E}[f(Y, X)|\mathcal{F}](\omega) = \mathbb{E}[f(Y(\omega), X)] \qquad \text{for } \mathbb{P}\text{-almost every } \omega \in \Omega.$$

3. (Revision of conditional expectations 3). Let X, Y, Z be random variables on a joint probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We define

$$\mathbb{E}[X|Y] := \mathbb{E}[X|\sigma(Y)].$$

Show the following statements:

a) If $X, Y \in \mathcal{L}^1$ are independent and identically distributed, then $\mathbb{P}$-almost surely,

$$\mathbb{E}[X|X + Y] = \frac{1}{2}(X + Y).$$

b) If Z is independent of the pair (X, Y), then $\mathbb{P}$-almost surely,

$$\mathbb{E}[X|Y, Z] = \mathbb{E}[X|Y].$$

Is this statement still true if we only assume that X and Z are independent?