Institute for Applied Mathematics
Winter term 2014/15
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“Markov Processes”, Problem Sheet 11.

Hand in solutions before Friday 23.01., 2 pm

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1. (Adjoint processes) Let p_t, $t \geq 0$, be the transition semigroup of a time-homogeneous Markov jump process on a *finite* state space S with generator $\mathcal{L}$. Let μ be a probability measure with full support on S.

a) Write down explicitly the adjoint $\mathcal{L}^*$ of $\mathcal{L}$ as an operator in $L^2(\mu)$. Prove that $\mathcal{L}^*$ is the generator of a Markov process if and only if μ is stationary w.r.t. $(p_t)_{t\geq 0}$.

b) Show that in this case, the Markov process generated by $\mathcal{L}^*$ has the transition semigroup p_t^*.

c) Give a probabilistic interpretation of this process when μ is the initial distribution.

2. (Ornstein-Uhlenbeck process) The transition semigroup of the Ornstein-Uhlenbeck process on $\mathbb{R}$ is given by

$$(p_t f)(x) \;=\; (2\pi)^{-1/2} \int f\left(e^{-t}x + \sqrt{1-e^{-2t}}\, y\right) e^{-y^2/2}\, dy \qquad \text{for } f \in \mathcal{F}_b(\mathbb{R}).$$

a) Show that the standard normal distribution γ is a stationary initial distribution.

b) Let L denote the generator on $L^2(\mathbb{R},\gamma)$. Show that $C^2_{\mathrm{pol}} \subset \mathrm{Dom}(L)$ and

$$(Lf)(x) \;=\; f''(x) - xf'(x) \qquad \text{for any } f \in C^2_{\mathrm{pol}}.$$

c) Show that p_t preserves polynomials. Hence conclude that C^2_{pol} is a core for the generator.

Remark. C^2_{pol} *denotes the space of continuous functions on* $\mathbb{R}$ *with at most polynomial growth at infinity.*

3. (Martingale problem for Feller processes) Let (p_t) be the transition function of a right-continuous time-homogeneous Markov process $((X_t)_{t\geq 0}, (P_x)_{x\in S})$ on a separable locally compact state space S such that

$$p_t\left(\hat{C}(S)\right) \subseteq \hat{C}(S) \qquad \forall\, t \geq 0.$$

a) Show that $(p_t)_{t\geq 0}$ induces a Feller semigroup, and (X_t, P_x) solves the martingale problem for the generator $(L, \mathrm{Dom}(L))$ for any $x \in S$.

b) Prove that for any $\alpha \geq 0$ and any $f \in \mathrm{Dom}(L)$,

$$M_t^{f,\alpha} \;=\; e^{-\alpha t} f(X_t) + \int_0^t e^{-\alpha s} \left(\alpha f - Lf\right)(X_s) ds$$

is a martingale.

4. (Differential operators as generators) Suppose that the generator of a Feller semigroup on $\mathbb{R}$ satisfies

$$(Lf)(x) \;=\; \sum_{n=0}^{m} a_n(x) \frac{d^n f}{dx^n}(x) \qquad \forall\, f \in C_0^\infty(\mathbb{R})$$

for some $m \in \mathbb{N}$ and coefficients $a_i \in C(\mathbb{R})$. Show that for any $x \in \mathbb{R}$,

$$a_0(x) \leq 0, \qquad a_2(x) \geq 0 \qquad \text{and} \qquad a_n(x) = 0 \;\;\forall\, n > 2.$$