Institute for Applied Mathematics
Winter term 2014/15
Andreas Eberle, Lisa Hartung

"Markov Processes", Problem Sheet 10.

Hand in solutions before Friday 09.01., 2 pm

We wish you a merry Christmas and a happy new year!

1. (Markov processes and martingales) Let $(p_{s,t})$ be a transition function on a measurable space $(S, \mathcal{B})$. Show that an adapted process (X_t, P) with state space $(S, \mathcal{B})$ is an $(\mathcal{F}_t)$-Markov process with transition function $(p_{s,t})$ if and only if $(p_{s,t}f)(X_s)$, $s \in [0, t]$, is an $(\mathcal{F}_s)$-martingale for any $t \geq 0$ and any $f \in \mathcal{F}_b(S)$.

2. (Uniform motion to the right) Consider a deterministic Markov process (X_t, P_x) on $\mathbb{R}$ given by $X_t = x + t$ P_x-almost surely.

a) Show that the transition semigroup $(P_t)_{t \geq 0}$ is strongly continuous both on $\hat{C}(\mathbb{R})$ and on $L^2(\mathbb{R}, dx)$.

b) Prove that the generator on $\hat{C}(\mathbb{R})$ is given by

$$Lf \;=\; f', \qquad \mathrm{Dom}(L) \;=\; \{f \in \hat{C}(\mathbb{R}) : f' \in \hat{C}(\mathbb{R})\}.$$

c) Show that the generator on $L^2(\mathbb{R}, dx)$ is given by

$$Lf \;=\; f', \qquad \mathrm{Dom}(L) \;=\; H^{1,2}(\mathbb{R}, dx).$$

3. (Brownian motion reflected at 0) Let $(B_t)_{t \geq 0}$ be a standard one-dimensional Brownian motion with transition density $p_t(x, y)$.

a) Show that $X_t = |B_t|$ is a Markov process with transition density

$$p_t^+(x, y) \;=\; p_t(x, y) + p_t(x, -y).$$

b) Prove that (X_t, P) solves the martingale problem for the operator $\mathcal{L}f = \frac{1}{2}f''$ with domain

$$\mathcal{A} \;=\; \{f \in C_b^2([0, \infty)) : f'(0) = 0\}.$$

Hint: Note that functions in $\mathcal{A}$ can be extended to symmetric functions in $C_b^2(\mathbb{R})$.

c) Construct another solution to the martingale problem for $\mathcal{L}$ with domain $C_0^\infty(0,\infty)$. In which sense do the generators of the two processes on $L^2(\mathbb{R}_+, dx)$ differ from each other ?

4. (Strongly continuous semigroups and resolvents)

a) State the defining properties of a strongly continuous contraction semigroup and a strongly continuous contraction resolvent on a Banach space E.

b) Prove that if (P_t) is a C_0 contraction semigroup then $G_\alpha f = \int_0^\infty e^{-\alpha t} P_t f\, dt$ defines a C_0 contraction resolvent.

c) Compute the resolvent of Brownian motion on $\hat{C}(\mathbb{R})$ explicitly.

5. (Immigration-death process) Particles in a population die independently with rate $\mu > 0$. In addition, immigrants arrive with rate $\lambda > 0$. Assume that the population consists initially of one particle.

a) Explain why the population size X_t can be modeled by a birth-death process with rates $b(n) = \lambda$ and $d(n) = n\mu$.

b) Show that the generating function $G(s,t) = \mathbb{E}(s^{X_t})$ is given by

$$G(s,t) \;=\; \{1 + (s-1)e^{-\mu t}\} \exp\left\{ \frac{\lambda}{\mu}(s-1)(1-e^{-\mu t}) \right\}$$

c) Deduce the limiting distribution of X_t as $t \to \infty$.

6. (Explosion, occupation times and stationary distributions for diffusions on $\mathbb{R}^n$) Consider a diffusion process (X_t, P_x) on $\mathbb{R}^n$ solving the local martingale problem for the generator

$$\mathcal{L}_t f \;=\; \frac{1}{2}\sum_{i,j=1}^n a_{i,j}(t,x) \frac{\partial^2 f}{\partial x_i\, \partial x_j} \;+\; \sum_{i=1}^n b_i(t,x) \frac{\partial f}{\partial x_i}, \qquad f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n).$$

We assume that the coefficients are continuous functions and $P_x[X_0 = x] = 1$.

a) Prove that the process is non-explosive if there exist finite constants c_1, c_2, r such that

$$\operatorname{tr} a(t,x) \;\le\; c_1\, |x|^2 \quad \text{and} \quad x \cdot b(t,x) \;\le\; c_2\, |x|^2 \qquad \text{for } |x| \ge r.$$

b) Now suppose that $\zeta = \infty$ almost surely, and that there exist $V \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$ and $\varepsilon, c \in \mathbb{R}_+$ such that $V \ge 0$ and

$$\frac{\partial V}{\partial t} + \mathcal{L}_t V \;\le\; \varepsilon + c 1_B \qquad \text{on } \mathbb{R}_+ \times \mathbb{R}^n,$$

where B is a ball in $\mathbb{R}^n$. Prove that

$$E\left[\frac{1}{t} \int_0^t 1_B(X_s)\, ds \right] \;\ge\; \frac{\varepsilon}{c} - \frac{V(0, x_0)}{ct}.$$

b) Conclude that if (X_t, P_x) is a time-homogenuous Markov process and the conditions above hold then there exists a stationary distribution.