Institute for Applied Mathematics
Winter term 2014/15
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"Markov Processes", Problem Sheet 9.

Hand in solutions before Friday 12.12., 2 pm
(post-box opposite to maths library)

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1. (Infinitesimal characterization of stationary distributions) Consider a time-homogeneous continuous time Markov chain $X_t = Y_{N_t}$ where (N_t) is a Poisson process with constant intensity $\lambda > 0$, and (Y_n) is an independent Markov chain with transition matrix π on a finite state space S.

a) Show that the transition function is given by

$$p_t(x,y) \;=\; P_x[X_t = y] \;=\; \exp(t\mathcal{L})\,(x,y),$$

where $\mathcal{L} = \lambda(\pi - I)$ and $\exp(t\mathcal{L})$ is the matrix exponential. Hence conclude that $(p_t)_{t\geq 0}$ satisfies the forward and backward equation

$$\frac{d}{dt}p_t \;=\; p_t\mathcal{L} \;=\; \mathcal{L}p_t \qquad \text{for } t \geq 0.$$

b) Prove that a probability measure μ on S is a stationary distribution if and only if

$$\sum_{x\in S} \mu(x)\,\mathcal{L}(x,y) \;=\; 0 \qquad \text{for any } y \in S.$$

c) Show that the transition matrices are self-adjoint in $L^2(\mu)$, i.e.,

$$\sum_{x\in S} f(x)\,(p_t g)(x)\;\mu(x) \;=\; \sum_{x\in S} (p_t f)(x)\,g(x)\;\mu(x) \qquad \text{for any } t\geq 0,\;\; f,g : S \to \mathbb{R},$$

if and only if the generator $\mathcal{L}$ satisfies the detailed balance condition w.r.t. μ. What does this mean for the process ?

2. (Simple exclusion process) Let $\mathbb{Z}^d_n = \mathbb{Z}^d/(n\mathbb{Z})^d$ denote a discrete d-dimensional torus. The simple exclusion process on $S = \{0,1\}^{\mathbb{Z}^d_n}$ is the Markov process with generator

$$(\mathcal{L}f)\,(\eta) \;=\; \frac{1}{2d}\sum_{x\in\mathbb{Z}^d_n}\;\sum_{y:|y-x|=1} 1_{\{\eta(x)=1,\eta(y)=0\}}\cdot\left(f\left(\eta^{x,y}\right) - f(\eta)\right),$$

where $\eta^{x,y}$ is the configuration obtained from η by exchanging the values at x and y. Show that any Bernoulli measure of type

$$\mu_p \;=\; \bigotimes_{x\in\mathbb{Z}^d_n} \nu_p\,, \qquad \nu_p(1) = p,\;\; \nu_p(0) = 1-p,$$

$p \in [0,1]$, is a stationary distribution. Why does this not contradict the fact that any irreducible Markov process on a finite state space has a unique stationary distribution ? *(You may assume the statements of Exercise 1).*

3. (Bounds for ergodic averages in the non-stationary case) Let $(X_n)_{n\in\mathbb{Z}_+}$ be a Markov chain on $(S,\mathcal{B})$ with transition kernel p and stationary distribution μ, and let

$$A_{b,n}f \;=\; \frac{1}{n}\sum_{i=b}^{b+n-1} f(X_i).$$

Assume that there are a distance d on S, $0<\alpha<1$ and $\bar{\sigma}\in\mathbb{R}_+$ such that

(A1) $\mathcal{W}_d^1(\nu p,\widetilde{\nu}p) \le \alpha \mathcal{W}_d^1(\nu,\widetilde{\nu}) \;\; \forall\, \nu,\widetilde{\nu}\in\mathcal{P}(S)$, and

(A2) $\mathrm{Var}_{p(x,\cdot)}(f) \le \bar{\sigma}^2 \|f\|^2_{Lip(d)} \;\; \forall\, x\in S,\; f: S\to\mathbb{R}$ Lipschitz.

Prove that under these assumptions the following bounds hold for any $b,n,k\ge 0$, $x\in S$, and for any Lipschitz continuous function $f: S\to\mathbb{R}$:

a) $\mathrm{Var}_x\left[f(X_n)\right] \;\le\; \sum_{k=0}^{n-1}\alpha^{2k}\,\bar{\sigma}^2\|f\|^2_{Lip(d)}$.

b) $\left|\mathrm{Cov}_x\left[f(X_n),f(X_{n+k})\right]\right| \;=\; \left|\mathrm{Cov}_x\left[f(X_n),(p^k f)(X_n)\right]\right| \;\le\; \frac{\alpha^k}{1-\alpha^2}\bar{\sigma}^2\|f\|^2_{Lip(d)}$.

c) $\mathrm{Var}_x\left[A_{b,n}f\right] \;\le\; \frac{1}{n}\frac{\bar{\sigma}^2}{(1-\alpha)^2}\|f\|^2_{Lip(d)}$.

d) $\left|E_x\left[A_{b,n}f\right]-\mu(f)\right| \;\le\; \frac{1}{n}\frac{\alpha^b}{1-\alpha}\int d(x,y)\,\mu(dy)\,\|f\|_{Lip(d)}$.

e) $E_x\left[|A_{b,n}f-\mu(f)|^2\right] \;\le\; \frac{1}{n}\frac{1}{(1-\alpha)^2}\left(\bar{\sigma}^2+\frac{1}{n}\alpha^{2b}(\int d(x,y)\,\mu(dy))^2\right)\|f\|^2_{Lip(d)}$.

4. (Succesful couplings and TV-convergence to equilibrium) Consider a Markov chain on $(S,\mathcal{B})$ with transition kernel p and stationary distribution μ. A coupling (X_n,Y_n) of the chains with initial distributions ν and μ respectively is called *succesful* if the coupling time

$$T \;=\; \inf\{n\ge 0 \,:\, X_n = Y_n \text{ for any } n\ge T\}$$

is almost surely finite. Show that a succesful coupling exists if and only if $||\nu p^n-\mu||_{TV}\to 0$ as $n\uparrow\infty$.