Institute for Applied Mathematics
Winter term 2014/15
Andreas Eberle, Lisa Hartung

"Markov Processes", Problem Sheet 8.

Hand in solutions before Friday 5.12., 2 pm
(post-box opposite to maths library)

---

1. (Asymptotic variances of ergodic averages) We consider a stationary Markov chain (X_n, P_μ) with state space $(S, \mathcal{B})$, transition kernel p, and initial distribution μ.

a) For $f \in \mathcal{L}^2(\mu)$ let $f_0 = f - \int f\, d\mu$, and

$$A_t f \;=\; \frac{1}{t}\sum_{i=0}^{t-1} f(X_i).$$

Prove (without assuming the CLT) that if $Gf_0 \equiv \sum_{k=0}^{\infty} p^k f_0$ converges in $\mathcal{L}^2(\mu)$ then

$$\lim_{t\to\infty} t\,\mathrm{Var}\,[A_t f] \;=\; 2(f_0, Gf_0)_{L^2(\mu)} - (f_0, f_0)_{L^2(\mu)} \;=\; \mathrm{Var}_\mu(f) + \sum_{k=1}^{\infty} \mathrm{Cov}_\mu(f, p^k f).$$

b) Let $S = \{1, 2\}$, and suppose that the transition rates are given by $p(1,1) = p(2,2) = p$ and $p(2,1) = p(1,2) = 1-p$ with $p \in (0,1)$. Show that the unique stationary distribution μ is given by $\mu(1) = \mu(2) = 1/2$ for all values of p. Now consider

$$S_n \;=\; A_n - B_n,$$

where A_n and B_n are, respectively, the number of visits to the states 1 and 2 during the first n steps. Show that $S_n/\sqrt{n}$ satisfies a central limit theorem, and calculate the limiting variance as a function $\sigma^2(p)$ of p. How does $\sigma^2(p)$ behave as p tends to 0 or 1? Can you explain it? What is the value of $\sigma^2(1/2)$? Could you have guessed it?

2. (Random Walks on $\mathbb{Z}_+$) Let $\delta \in (0,1)$. We consider a random walk on the nonnegative integers with transition probabilities

$$p(x,y) = \begin{cases} \frac{1}{2} & \text{for } x = y \geq 0, \\ \frac{1+\delta}{4} & \text{for } y = x+1,\ x \geq 1, \\ \frac{1-\delta}{4} & \text{for } y = x-1,\ x \geq 1 \\ \frac{1}{2} & \text{for } x = 0,\ y = 1. \end{cases}$$

a) Find the stationary distribution $\mu(x)$ explicitly.

b) If $f(x)$ is a function on $\mathbb{Z}_+$ with compact support, solve the equation $-\mathcal{L}g = f$ explicitly (e.g. by the variation of constants ansatz $g = uh$ where h is a nontrivial solution of $\mathcal{L}h = 0$). Show that a solution g either grows exponentially at infinity or is a constant for large x.

c) Show that there is a solution g that is a constant for large x if and only if $\int f \, d\mu = 0$. What can you say about the asymptotic variance and the central limit theorem for $\sum_{j=0}^{n-1} f(X_j)$ for such functions f ?

3. (Equivalent characterizations of ergodicity for Markov processes)
We consider a canonical right-continuous Markov process $((X_t)_{t\geq 0}, P_x)$ with state space $(S, \mathcal{B})$, transition semigroup $(p_t)_{t\geq 0}$, and stationary initial distribution μ. Show that the following nine conditions are all equivalent:

(i) P_μ is ergodic.

(ii) $\frac{1}{t}\int_0^t f(X_s)ds \to \int f d\mu$ P_μ-a.s. and in $L^2(P_\mu)$, for any $f \in \mathcal{L}^2(\mu)$.

(iii) $\mathrm{Var}_{P_\mu}\left[\frac{1}{t}\int_0^t f(X_s)\, ds\right] \to 0$ as $t \uparrow \infty$ for any $f \in \mathcal{L}^2(\mu)$.

(iv) $\frac{1}{t}\int_0^t \mathrm{Cov}_{P_\mu}\left[f(X_0), f(X_s)\right]\, ds \to 0$ as $t \uparrow \infty$ for any $f \in \mathcal{L}^2(\mu)$.

(v) $\frac{1}{t}\int_0^t P_\mu[X_0 \in B, X_s \in C]ds \to \mu(B)\mu(C)$ for any $B, C \in \mathcal{B}$.

(vi) $\frac{1}{t}\int_0^t p_s(x, B)ds \to \mu(B)$ $\mu-$a.e. for any $B \in \mathcal{B}$.

(vii) $P_x[T_B < \infty] > 0$ μ-a.e. for any $B \in \mathcal{B}$ such that $\mu(B) > 0$.

(viii) Every set $B \in \mathcal{B}$ such that $p_t 1_B = 1_B$ μ-a.e. for any $t \geq 0$ satisfies $\mu(B) \in \{0, 1\}$.

(ix) Every function $h \in \mathcal{L}^2(\mu)$ such that $p_t h = h$ μ-a.e. $\forall t \geq 0$ is almost surely constant.

4. (Structure of invariant measures) Let p be a transition kernel on $(S, \mathcal{B})$ and let

$$\mathcal{S}(p) = \{\mu \in \mathcal{P}(S) : \mu = \mu p\}.$$

a) Show that $\mathcal{S}(p)$ is convex.

b) Prove that $\mu \in \mathcal{S}(p)$ is extremal if and only if every set $B \in \mathcal{B}$ such that $p 1_B = 1_B$ μ-a.e. satisfies $\mu(B) \in \{0, 1\}$.

c*) Show that every $\mu \in \mathcal{S}(p)$ is a convex combination of extremals.
(Hint: You may use part c) of Exercise 3 of the previous problem sheet.)