Institute for Applied Mathematics
Winter term 2014/15
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"Markov Processes", Problem Sheet 6.

Hand in solutions before Friday 21.11, 2 pm
(post-box opposite to maths library)

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1. (Stationary processes) Let $((X_n)_{n\in\mathbb{Z}_+}, P)$ be a canonical stationary process with state space S, and let $\mathcal{J} = \{A : A = \Theta^{-1}(A)\}$ be the σ-algebra of shift invariant events.

a) Prove that for $F : \Omega \to \mathbb{R}$ the following two properties are equivalent:

(i) F is $\mathcal{J}$- measurable.

(ii) $F = F \circ \Theta$.

Conclude that P is ergodic if and only if any shift-invariant function $F : \Omega \to \mathbb{R}$ is P-almost surely constant.

b) Show that $((X_n)_{n\in\mathbb{Z}_+}, P)$ can be extended to a two-sided stationary process $((\hat{X}_n)_{n\in\mathbb{Z}}, \hat{P})$ on $\hat{\Omega} = S^{\mathbb{Z}}$ such that $\hat{X}_{0:\infty} \sim X_{0:\infty}$.
Hint: You may apply Kolmogorov's extension theorem.

c) Show that for any measurable function $G : S^{\mathbb{Z}_+} \to \mathbb{R}$, the process $(G(X_{n:\infty}))_{n\in\mathbb{Z}_+}$ is again stationary under P.

2. (Conductance and lower bounds for mixing times) Let p be a transition kernel on $(S, \mathcal{B})$ with stationary distribution μ. For sets $A, B \in \mathcal{B}$ with $\mu(A) > 0$, the *equilibrium flow* $Q(A, B)$ from A to B is defined by

$$Q(A, B) \;=\; (\mu \otimes p)(A \times B) \;=\; \int_A \mu(dx)\, p(x, B),$$

and the *conductance* of A is given by

$$\Phi(A) \;=\; \frac{Q(A, A^C)}{\mu(A)}\,.$$

The *bottleneck ratio (isoperimetric constant)* Φ_* is defined as

$$\Phi_* = \min_{A:\mu(A)\leq 1/2} \Phi(A).$$

Let $\mu_A(B) = \mu(B|A)$ denote the conditioned measure on A.

a) Show that for any $A \in \mathcal{B}$ with $\mu(A) > 0$,

$$\|\mu_A p - \mu_A\|_{TV} \;=\; (\mu_A p)(A^C) \;=\; \Phi(A).$$

Hint: Prove first that

(i) $(\mu_A p)(B) - \mu_A(B) \leq 0$ *for any measurable* $B \subseteq A$, *and*

(ii) $(\mu_A p)(B) - \mu_A(B) = (\mu_A p)(B) \geq 0$ *for any measurable* $B \subseteq A^C$.

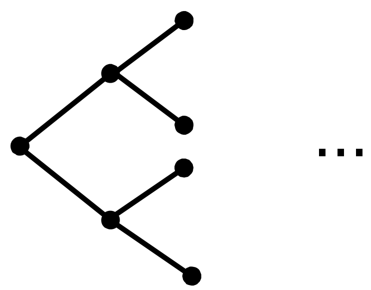
b) Conclude that

$$\|\mu_A - \mu\|_{TV} \leq t\Phi(A) + \|\mu_A p^t - \mu\|_{TV} \qquad \text{for any } t \in \mathbb{Z}_+.$$

c) Hence prove the lower bound

$$t_{mix}\left(\frac{1}{4}\right) \;\geq\; \frac{1}{4\Phi_*}.$$

3. (Lazy random walk on a binary tree)



Consider the lazy random walk with resting probability $p(x,x) = 1/2$ on a binary tree of depth k. Let $m = 2^{k+1} - 1$ denote the number of vertices. Prove that:

a) $t_{mix}(1/4) \;=\; O(m)$.

b) $t_{mix}(1/4) \;=\; \Omega(m)$.

Hint: You may assume the conductance bound from the previous exercise!